POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

(Approved by AICTE and affiliated to Anna University,Coimbatore.)

107/1-B,POOSARIPATTI,POLLACHI-642 205.

QUESTION BANK

MATHEMATICS – 1

THREE DIMENSIONAL ANALYTICAL GEOMETRY

PART- A

1. Find the equation of the sphere whose centre is (2, 3, 1) and radius is 5 units.

   Solution:

   We know that the equation of a sphere whose centre is (a,b,c) and radius r is,

   $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ -- (1)

   Given: centre is (2, 3, 1) and radius is 5 units.

   Here a=2, b=-3, c=1 and r=5

   Substituting these values in equation (1) we get

   $$(x-2)^2 + (y-(-3))^2 + (z-1)^2 = 5^2$$

   $$(x-2)^2 + (y+3)^2 + (z-1)^2 = 5^2$$

   ie., $x^2 + 4 - 4x + y^2 + 9 + 6y + z^2 + 1 - 2z = 25$

   ie., $\mathbf{x^2 + y^2 + z^2 - 4x + 6y - 2z - 11 = 0.}$

   Which is the required equation of sphere.

2. Find the centre and radius of the sphere $x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$.

   Solution:

   Given : $x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$.

   We know that the general equation of a sphere is $\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.}$

Here	2u=2	2v=-4	2w=-6	d=5
ie.,	u=1	v=-2	w=-3	

   ∴ Centre : (-u,-v,-w)= (-1,2,3)

   Radius = $\sqrt{(u^2 + v^2 + w^2 - d)}$

   $= \sqrt{((-1)^2 + 2^2 + 3^2 - 5)}$

$=\sqrt{(1+4+9-5)}$

$=\sqrt{9}$

$=3$

Hence the centre of the given sphere is (-1,2,3) and radius is 3 units

3. Find the centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$.
   Solution:

   We know that the general equation of a sphere is $\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$.

   Given: $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$. --------------------(1)
   Dividing equation (1) by 2 we get,
   $x^2 + y^2 + z^2 - x + 2y + z + 3/2 = 0$.

Here	$2u=-1$	$2v=2$	$2w=1$	$d=3/2$
ie.,	$u=-1/2$	$v=1$	$w=1/2$	

   $\therefore$ Centre : $(-u,-v,-w) = (1/2, -1, -1/2)$

   Radius = $\mathbf{\sqrt{(u^2 + v^2 + w^2 - d)}}$

   $= \sqrt{(1/2)^2 + (-1)^2 + (-1/2)^2 - 5}$

   $= \sqrt{1/4 + 1 + 1/4 - 3/2}$

   $=0.$

   Hence the centre of the given sphere is (1/2, -1, -1/2) and radius is 0 units.

4. Find the equation of the sphere whose centre is (1,1,1) and which passes through the point (2,0,3)
   Solution:

   We know that the general equation of a sphere is $\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$**.-------(1)**

   Given : Centre=(1,1,1)

   ie., $-u=1$ $-v=1$ $-w=1$

   ie., $u=-1$ $v=-1$ $w=-1$

$\therefore$ equation (1) becomes, $x^2 + y^2 + z^2 - 2x - 2y - 2z + d = 0$. -----------------------(2)

Equation (2) passes through the point (2,0,3)

$\therefore$ equation (2) becomes, $2^2 + 0^2 + 3^2 - 2(2) - 2(0) - 2(3) + d = 0$.

$4+0+9-4-0-6+d=0$

$3+d=0$

$d=-3$

substitute d= -3 in equation (2) we get,

$x^2 + y^2 + z^2 - 2x - 2y - 2z - 3 = 0$, which is the required equation of the sphere.

5. Find the equation of the sphere of centre at (1,2,3) and touching a plane at (2,1,3).

Solution:

Radius of the sphere = the distance between the centre of the sphere and the point,which the plane touches the sphere

$= \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$

$= \sqrt{(2-1)^2+(1-2)^2+(3-3)^2}$

$= \sqrt{1+1+0}$

$r = \sqrt{2}$

We know that the equation of a sphere whose centre is (a,b,c) and radius r is,

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

$(x-1)^2 + (y-2)^2 + (z-3)^2 = (\sqrt{2})^2$

$x^2-2x+1+y^2-4y+4+z^2-6z+9=2$

$x^2+y^2+z^2-2x-4y-6z+12=0$, which is the required equation of the sphere.

6. Find the equation of the sphere on the join of (1, -1, 1) and (-3, 4, 5) as diameter.

Solution:

We know that the equation of a sphere whose diameter is the join of (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$ --------------------(1)

Here $x_1=1$, $y_1=-1$, $z_1=-1$

$x_2=-3$, $y_2=4$, $z_2=5$

Equation (1) becomes,

$(x-1)(x+3) + (y+1)(y-4) + (z+1)(z-5) = 0$

$x^2+2x-3+y^2-3y-4+z^2-4z-5=0$

$x^2+y^2+z^2+2x-3y-4z-12=0$, which is the required equation of the sphere.

7. Find the centre and radius of the sphere $a(x^2+y^2+z^2) +2ux+2vy+2wz+d=0$

Solution:

Given : $a(x^2+y^2+z^2) +2ux+2vy+2wz+d=0$ ---------------(1)

Dividing equation (1) by 'a' we get,

$(x^2+y^2+z^2) +2ux/a+2vy/a+2wz/a+d/a=0$

Centre = [-1/2 co.eff of x, -1/2 co.eff of y, -1/2 co.eff of z]
= [-u/a, -v/a, -w/a]

Radius $=\sqrt{[(u^2 + v^2 + w^2)/a^2 - d/a]}$

8. Find the equation of the sphere which passed through the point (1, -2, 3) and the circle $z=0$, $x^2+y^2+z^2 -9 = 0$

Solution:

Any sphere through the given circle is $(x^2+y^2+z^2 -9) + \lambda z =0$ -----------(1)

If if passes through the given point (1, -2, 3) we get,

$(1+4+9-9) + \lambda(3) = 0$

$3\lambda = -5$

$\Lambda = -5/3$ ---------------------(2)

Substitute equation (2) in equation (1) we get,

$(x^2+y^2+z^2 -9) -5/3\, z =0$

$3(x^2+y^2+z^2) – 5z - 27=0$, which is the required equation of the sphere.

9. Find the equation of the sphere with centre (0,1,0) and touching the plane $x-2y+2z+5=0$.

Solution:

Since the plane touches the sphere,
Radius = Perpendicular distance from centre.

$$= \pm\left\{\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}\right\}$$ (a=1, b=-2,c=2 & d=5)

$$=\pm\left\{\frac{0-2+0+5}{\sqrt{1+4+4}}\right\}$$

$=\pm 3/3$

Radius $=\pm 1$

10. Find the equation of the tangent plane at (1,1,1) to $x^2+y^2+z^2-3=0$

Solution:

We know that the equation of tangent plane to the sphere at (x_1,y_1,z_1) is,

$$\mathbf{x\,x_1 + y\,y_1 + z\,z_1 + u(x+x_1) + v(y+y_1) + w(z+z_1) + d = 0} \quad \text{--------(1)}$$

$x(1) + y(1) + z(1) +0+0+0 -3= 0$ ($\because x_1=1, y_1=1, z_1=1$ and $d=-3$)

$x + y + z =3$, which is the required equation of tangent plane.

11. What is the condition for the plane Ax+By+Cz+D=0 may touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Solution:

$(Au+Bv+Cw-D)^2 = (A^2+B^2+C^2)(u^2+v^2+w^2-d)$

12. Find the equation of the sphere with centre at (1,5,6) and which touches the Z axis.

Solution :

The required sphere touches the Z axis at the point P(0,0,6)

$\therefore$ Radius = Distance between C(1,5,6) and P(0,0,6)

$= \sqrt{(1-0)^2+(5-0)^2+0}$

$=\sqrt{1+25}$

$=\sqrt{26}$

$\therefore$ Equation of sphere with centre at (1,5,6) and $r=\sqrt{26}$ is

$(x-1)^2 + (y-5)^2 + (z-6)^2 = (\sqrt{26})^2$

$\Rightarrow x^2-2x+1+y^2-10y+25+z^2-12z+36=26$

$\Rightarrow$ **$x^2+y^2+z^2-2x-10y-12z+36=0$, which is the required equation of the sphere.**

13. If the radius of $x^2+y^2+z^2-4x-4y+2z+k=0$ is 5 units, find k.

Solution:

$r = \sqrt{u^2+v^2+w^2-d}$

⇨ $5 = \sqrt{4+4+1-k}$

⇨ $5 = \sqrt{9-k}$

⇨ $25 = 9-k$

⇨ $25-9 = -k$

⇨ $k = -16$

14. Find the equation of the sphere concentric with $x^2+y^2+z^2+2x+2y+2z-1=0$ nd having radius as 3 units.

Solution:

Centre of the given sphere : (-1,-1,-1)

Since the required sphere is concentric with the given sphere, their centres will be the same.

We know that the equation of a sphere whose centre is (a,b,c) and radius r is,

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

∴ Equation of required sphere with centre at (-1,-1,-1) and radius 3 is

$$(x+1)^2+(y+1)^2+(z+1=3^2$$

$$x^2+2x+1+y^2+2y+1+z^2+2z+1=9$$

$x^2+y^2+z^2+2x+2y+2z-6=0$, which is the required equation of the sphere.

15. (2,3,4) is one end of the diameter of sphere $x^2+y^2+z^2+2x-2y+4z-1=0$, find the other end.

Solution:

Centre : (1,1,-2)

One of the ends of the diameter : (2,3,4)

Let the other end of the diameter : (a,b,c)

The centre C of the sphere is the midpoint of the line joining the extremities (A and B) of the diameter.

ie., $(1,1,-2) = \left(\frac{2+a}{2},\frac{3+b}{2},\frac{4+c}{2}\right)$

⇨ $\frac{2+a}{2}=1$; $\frac{3+b}{2}=1$; $\frac{4+c}{2}=-2$

⇨ $2+a = 2$; $3+b=2$; $4+c=-2$

⇨ $a=0$; $b=-1$; $c=-8$

The other end is B(0,-1,-8)

16. Find the equation of the sphere passing through (0,0,0) , (1,0,0), (0,1,0) and (0,0,1)

Solution:

Let the equation of the required sphere be $\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$.------(1)

∵ (1) passes through (0,0,0), d=0

∵ (1) passes through (1,0,0), we have u=-1/2

∵ (1) passes through (0,1,0), we have v=-1/2

∵ (1) passes through (0,0,1), we have w=-1/2

Substituting u,v,w and d in (1), we get

$x^2 + y^2 + z^2 + 2(-1/2)x + 2(-1/2)y + 2(-1/2)z + 0 = 0$

=> $\mathbf{x^2 + y^2 + z^2 - x - y - z = 0}$**, which is the required equation of the sphere.**

17. What is the condition for orthogonally of two spheres?

Solution:

S: $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

S_1: $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$

The condition for orthogonally of two spheres(S & S_1) is $\mathbf{2u\, u_1 + 2v\, v_1 + 2w\, w_1 = d + d_1}$

18. Define orthogonal spheres.

Solution:

Two spheres are said to be orthogonal, if the tangent planes at the point of intersection are at right angles.

19. Define a great circle.

Solution:

The section of a sphere by a plane passing through its centre is called a great circle.

20. Show that the plane 2x-2y+z+12=0 touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$.

Solution:

Radius = $\mathbf{\sqrt{(u^2 + v^2 + w^2 - d)}}$

$= \sqrt{1 + 4 + 1 + 3}$

$= \sqrt{9}$

$= 3$

Perpendicular distance from the centre c(1,2,-1) to the plane

2x-2y+z+12=0

$$=\pm\left\{\frac{\mathbf{2-4-1+12}}{\sqrt{(\mathbf{4+4+1})}}\right\}$$

$= 9/3$

$=3$

Thus the plane touches the sphere.

PART - B

1. Find the equation of the sphere which passes through the points (2, 0, 0), (0, 2, 0) and (0, 0, 2) and which has its radius as small as possible.

2. Find the equations of the tangent planes to the sphere $x^2+y^2+z^2+2x-4y+6z-7=0$ which passes through the line $6x-3y-23=0=3z+2$.

3. Find the equations of sphere which pass through the circle $x+2y+3z=8$, $x^2+y^2+z^2-2x-4y=0$ and touch the plane $4x+3y=25$.

4. Find the center and radius of the circle $x^2+y^2+z^2-8x+4y+8z-45=0$ and $x-2y+z-3=0$.

5. Show that the spheres $x^2+y^2+z^2+6y+2z+8=0$ and $x^2+y^2+z^2+6x+8y+4z+20=0$ cut orthogonally. Find their plane of intersection. Also prove that this plane is perpendicular to the line joining the center.

6. Find the equation of the sphere passing through the points (0, 3, 0), (−2,−1,−4) and cutting orthogonally the two spheres $x^2+y^2+z^2+x-3z-2=0$ and $2x^2+2y^2+2z^2+x+3y+4=0$.

7. Find the equation of the cone whose vertex is (1, 2, 3) and guiding curve is the circle $x^2+y^2+z^2=4$, $x+y+z=1$.

8. Prove that $9x^2+9y^2-4z^2+12yz-6zx+54z-81=0$ represents a cone. Find also its vertex.

9. Find the equation of the cylinder whose axis is $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1$, $z=3$.

10. Find the equation of the right circular cylinder passing through A (3, 0, 0) and having the axis $x-2=z$, $y=0$.

11. Find the equation of the right circular cone generated by the straight line drawn from the origin to cut the circle through the three points (1, 2, 2); (2, 1, -2) and (2, -2, 1).

12. Determine the angle between the lines of intersection of the plane $x-3y+z = 0$ and the cone $x^2-5y^2+z^2 = 0$.

13. Find the equation to the cones with vertex at the origin and which passes through the curve $ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

14. Find the equation of the cylinder whose generators are parallel to the z-axis and which passes through the curve of intersection of $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$.

15. Find the angle between the lines of section of the plane $3x + y + 5z = 0$ and the cone $6yz - 2zx + 5xy = 0$.