POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY

(Approved by AICTE and affiliated to Anna University, Chennai.)

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UNIT – III DIFFERENTIAL CALCULUS

PART – A

1. Find the radius of curvature at the point (r,ψ) on the curve s = c log secψ

   The given curve is s = c log secψ

   $\rho = \frac{ds}{d\psi}$

   $= \frac{c}{\sec\Psi} * \sec\Psi \tan\Psi$ =c tanψ

2. Find the radius of curvature of the curve $y = e^{x}$ at the point, where it cuts the y – axis

   To find the point where the given curve cuts y-axis, put x=0

   $\therefore$ $y = e^{0} = 1$

   Thus , the curve cuts the y-axis at (0, 1).

   $y = e^x \Rightarrow y_1 = e^x, y_2 = e^x$

   The value of y_1 and y_2 at (0, 1) is 1.

   $\therefore \rho = \frac{\left(1+y_1^{2}\right)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{y_2} = 2^{\frac{3}{2}}$

3. Find the radius of curvature of the curve given by the equations x = 5t, y= 5log(sect) at t = 1.

   The given equations are x = 5t, y= 5log(sect)

   $x_1 = 5, y_1 = 5 * \frac{1}{\sec t} * \sec t \tan t = 5\tan t$

   $\therefore \rho = \frac{\left(x_1^{2}+y_1^{2}\right)^{3/2}}{x_1y_2 - y_1x_2} = \frac{\left(5^2+5^2\tan^2 t\right)^{3/2}}{25\sec^2 t - 0} = \frac{(5)^3(1+\tan^2 t)^{3/2}}{25\sec^2 t}$ =5 sect

   $\therefore$ ρ (t = 1) = 5 sec(1)

4. Find the envelope of the family of straight lines $y = mx + \frac{1}{m}$

   Given $y = mx + \frac{1}{m}$

   $my = m^2x + 1$

   $m^2x - my + 1 = 0$. This is quadratic in the parameter 'm'. $\therefore$ envelope is $B^2-4AC = 0$.

   Here A =x, B= -y , C= 1. $\therefore$ Envelope is $y^2 - 4x = 0$.

5. Find ρ for the curve x = acos θ, y = a sin θ at θ.

Given x = acos θ, y = a sin θ

$$\frac{dx}{d\vartheta} = -a\sin\vartheta, \frac{dy}{d\theta} = a\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$$

$$\frac{d^2y}{d^2x} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\cot\theta)$$

$$= \text{cosec}^2\theta * \frac{d\theta}{dx} = \text{cosec}^2\theta * \frac{1}{a\sin\theta} = \frac{1}{a\sin^3\theta}$$

$$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{d^2x}} = \frac{\left\{1+\left(\cot^2\theta\right)^{3/2}\right\}}{\frac{1}{a\sin^3\theta}} = \text{cosec}^3\theta * a\sin^3\theta = a$$

6. What is the radius of curvature at (3, 4) on $x^2 + y^2 = 25$?

The given curve is a circle of radius 5 unit. We know that the radius of curvature of a circle is equal to the radius of the given circle. $\therefore$ $\rho = 5$.

7. Find the envelope of x cos α + y sinα = 1 where α is the parameter.

Given x cos α + y sinα = 1 $\rightarrow$ (1)

Partially differentiating (1) with respect to 'α' we get

-x sinα + y cos α = 0 $\rightarrow$ (2)

$(1)^2 + (2)^2 \Rightarrow$

$(x\cos\alpha + y\sin\alpha)^2 + (y\cos\alpha - x\sin\alpha)^2 = 1$

$x^2\cos^2\alpha + y^2\sin^2\alpha + 2xy\sin\alpha\cos\alpha + x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy\sin\alpha\cos\alpha = 1$

i.e. $x^2 + y^2 = 1$ which is the required equation of the envelope.

8. Write the formula for radius of curvature in cartesian form.

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

9. Find the radius of curvature at x = 1 on y = $\frac{\log x}{x}$

y ' = $\frac{1-\log x}{x^2}$, y '' = $\frac{2\log x-3}{x^3}$

(y ')$_{x=1}$ =1, (y '')$_{x=1}$ = -3

ρ = $\frac{(1+1)^{3/2}}{-3}$

= (2√2)/ 3

10. Find the radius of curvature for the curve y = $\frac{x^3-a^3}{a^2}$ at any point (x , y)

Given $a^2 y = x^3 - a^3$

Differentiating with respect to 'x' we get

$a^2\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{a^2}$, $\frac{d^2y}{dx^2}$ = $\frac{6x}{a^2}$

ρ = $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ = $\frac{\left[1+\left(\frac{3x^2}{a^2}\right)^2\right]^{3/2}}{\frac{6x}{a^2}}$ = $\frac{\left[a^4+\left(9x^4\right)\right]^{3/2}}{\left(6a^4x\right)}$

11. Find the envelope of ty-x = at ^{2}, t being the parameter.

Given ty-x = at ^{2} $\rightarrow$ (1)

Partially differentiating (1) with respect to 't',

y = 2at $\Rightarrow$ t = y / 2a

Substituting in (1), $\frac{y^2}{2a} - x = a\left(\frac{y}{2a}\right)^2$

$\frac{y^2}{2a} - x = ay^2/4a^2$. Simplifying we get y^2 = 4ax which is the required envelope of the given curve.

12. Find 'ρ' for the curve y = c cosh(x/c) at any point (x, y)

$\frac{dy}{dx} = \frac{1}{c}\sinh\frac{x}{c} * c$

$\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}$

ρ = $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ = $\frac{\left[1+\sinh^2\frac{x}{c}\right]^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$ = c $\frac{\left[\cosh^2\frac{x}{c}\right]^{3/2}}{\cosh\frac{x}{c}}$ = c $\frac{\cosh^3\frac{x}{c}}{\cosh\frac{x}{c}}$ =c cosh(x/c)

13. Find the radius of curvature at (3, 10) on the curve xy = 30

Differentiating with respect to 'x' we get $x\frac{dy}{dx}+y=0$

$\frac{dy}{dx}=-y/x$, $\{\frac{dy}{dx}\}_{(3,\,10)} = -10/3$

$$\frac{d^2y}{dx^2}=-\frac{x\frac{dy}{dx}-y}{x^2}$$

$$\frac{d^2y}{dx^2}(3,10)=\frac{20}{9}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{-10}{3}\right)^2\right]^{3/2}}{\frac{20}{9}} = \frac{\left(\frac{109}{9}\right)^{3/2}}{\frac{20}{9}} = \frac{9}{20}\frac{(109)^{3/2}}{9^{3/2}}$$

14. Find the envelope of the family of circles $(x-\alpha)^2+y^2=4\alpha$, where α is the parameter.

Given $(x-\alpha)^2+y^2=4\alpha \rightarrow$ (1)

Differentiating with respect to 'α' we get

$2(x-\alpha)(-1) = 4 \Rightarrow \alpha - x = 2 \Rightarrow \alpha = 2 + x$

Substituting in (1),

$(x-2-x)^2+y^2=4(2+x)$

$4+y^2=8+4x \Rightarrow y^2=4+4x=4(1+x)$

15. Find the envelope of the family of curves $y=\alpha x+\sqrt{a^2\alpha^2+b^2}$, where α being the parameter.

The given equation can be written as $(y-\alpha x)^2 = a^2\alpha^2+b^2$

$y^2-2\alpha xy+\alpha^2x^2=a^2\alpha^2+b^2$

$(x^2-a^2)\alpha^2-2xy\alpha+(y^2-b^2) = 0$,

which is quadratic in α.

The envelope is given by $B^2-4AC=0$

$4b^2x^2+4a^2y^2=4a^2b^2$ Dividing throughout by $4a^2b^2$,

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is an ellipse.

16. Find ρ for the curve y = 4 sinx – sin2x at x = 90®

Given y = 4 sinx – sin2x

$\frac{dy}{dx}=$ 4cosx - 2cos2x , $\frac{d^2y}{dx^2}=$ 4 sinx – 4 sin2x

$$\left[\frac{dy}{dx}\right]_{x=\pi/2} = 2, \quad \left[\frac{d^2y}{dx^2}\right]_{x=\pi/2} = -4$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{[1+4]^{3/2}}{-4}$$

$=(5\sqrt{5}) / 4$

17. Find the radius of curvature at y=2a on the curve $y^2 = 4ax$

$y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$ and $\frac{d^2y}{dx^2} = \frac{-2a}{y^2}\left(\frac{dy}{dx}\right)$

At y = 2a, $y_1 = 1$, $y_2 = -1 / 2a$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= - \frac{[1+1]^{3/2}}{\frac{1}{2a}} = -4a\sqrt{2}$$

18. What is the curvature of a (i) circle (ii) straight line
    (i) Curvature of a circle = 1/ r where r is the radius of the circle.
    (ii) Curvature of a straight line is zero

19. For the curve $x^2 = 2c(y-x)$, find the radius of curvature at (0,c)

$y - c = \frac{x^2}{2c} \Rightarrow \frac{dy}{dx} = \frac{x}{c}$ and $\frac{d^2y}{dx^2} = \frac{1}{c}$

At (0,c), $y_1 = 1$, $y_2 = 1 / c$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \text{ Diff}$$

20. Define evolute and involute.

The locus of centre of curvature at any point on a curve is called its evolute and the curve itself is called an involute of the evolute

PART – B

1. Find the radius of curvature at the point ($\frac{3a}{2}, \frac{3a}{2}$) on the curve $x^3 + y^3 = 3axy$
2. Find the equation of the circle of curvature of the curve √x + √y = √a at $(\frac{a}{4}, \frac{a}{4})$
3. Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at the point (3, 6)
4. Find the radius of curvature of the curve $r^n = a^n \cos n\vartheta$ at any point (r, θ). Hence prove that the radius of curvature of the lemniscate $r^2 = a^2 \cos 2\vartheta$ is $\frac{a^2}{3r}$
5. Find the evolute of the parabola $y^2 = 4ax$
6. Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
7. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
8. Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$
9. Find the evolute of the rectangular hyperbola xy= c^2
10. Show that the evolute of the cycloid x = a (θ - sin θ), y = a (1-cosθ) is another cycloid.
11. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation $a^2 + b^2 = c^2$
12. Find the envelope of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the parameters a and b are connected by the relation $a + b = c$
13. Find the evolute of the parabola $y^2 = 4ax$, considering it as the envelope of its normals.
14. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, considering it as the envelope of its normals.
15. Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,treating it as the envelope of its normals.