POLLACHI INSTITUTE OF ENGINEERING AND TECHNOLOGY
(Approved by AICTE and Affiliated to Anna University)
sky is the limit

## DEPARTMENT OF SCIENCE AND HUMANITIES

## REGULATION 2021

## I YEAR /II SEM

# MA3251 STATISTICS & NUMERICAL METHODS

MA3251

# STATISTICS AND NUMERICAL METHODS

UNIT 1: TESTING OF HYPOTHESIS

UNIT 2: DESIGN OF EXPERIMENTS

UNIT 3: SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

UNIT 4: INTERPOLATION, NUMERICAL DIFFERENTIATION AND NUMERICAL INTEGRATION

UNIT 5: NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

# Unit - I

**Statistics** : It is the science which deals with collection, presentation, analysis and interpretation of numerical data.

**Population** : It is a set of objects. The number of elements in this population may be finite ~~and~~ or infinite.

**Sample** : The part selected from the population is called sampling.

**Symbols** :

| Population parameters | Sample Statistics |
|---|---|
| population size $= N$ | Sample size $= n$ |
| population mean $= \mu$ | Sample mean $= \bar{x}$ |
| population Standard deviation $= \sigma$ | Sample standard deviation $= S$ |
| population variance $= \sigma^2$ | Sample variance $= S^2$ |

**Standard Error**: Standard deviation of sampling distribution is called standard error. It is abbreviated as S.E

**Hypothesis** : It is a statement about population parameter

**Null Hypothesis** $H_0$ : "There is no significant difference between the population parameter and sample statistic

**Alternate Hypothesis** $H_1$ : It is complementary to null hyp.

Example Suppose $H_0 : \mu = 1600$

3 possibilities for $H_1$

$H_1 : \mu \neq 1600$ (Two-tail Test)

$H_1 : \mu > 1600$ (Right tail Test)

$H_1 : \mu < 1600$ (Left Tail Test)

Important Note: Sampling distribution follows normal distribution when n is large

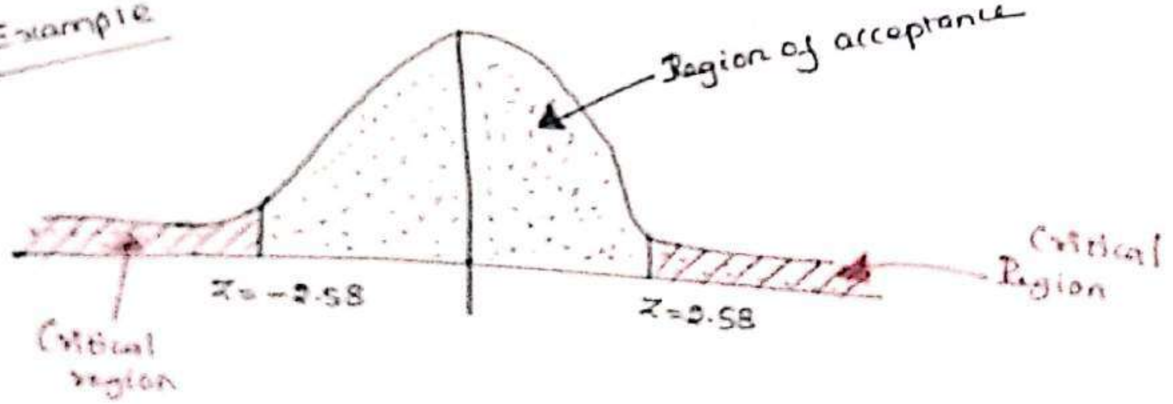
Normal curve is "Bell shaped"

∴ We can study using Normal distribution properties

Critical region: A region which amounts to rejection of $H_0$ is critical region. It also may be called as region of ~~acc~~ rejection.

Region of Acceptance: It is the region of complement to the region of rejection under normal curve

Example




Type I Error: Reject $H_0$ when $H_0$ is TRUE

Type II Error: Accept $H_0$ when $H_0$ is FALSE

# t - test for Single mean

Given a random Sample of size n $(n < 30)$ with sample mean $\bar{x}$, and the population Standard deviation is not known and we want to test "whether the population mean has a specified value"

Then we apply t-test.

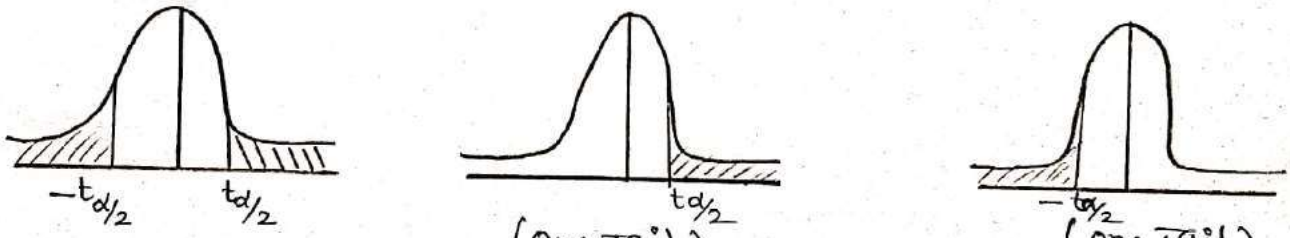
## Working Procedure :-

1) Set Null hypothesis $H_0 : \mu = \mu_0$
2) Set Alternate hypothesis $H_1 : \mu \neq \mu_0$ (Two tailed test)

   $H_1 : \mu > \mu_0$ (One tail test) - Right

   $H_1 : \mu < \mu_0$ (One tail test) - Left.

3) degrees of freedom $v = n-1$
4) Level of significance : $\alpha$
5) Critical region :




(Two-tailed Test) (One Tail Right) (One Tail Left)

6) Test statistic $t = \dfrac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)}$

   Here $\bar{x} \rightarrow$ Sample mean,

   $n \rightarrow$ Sample size

   $s \rightarrow$ Sample Standard deviation

   $$s^2 = \frac{(\Sigma x^2)}{n} - (\bar{x})^2$$

4) Conclusion :-

(a) If $-t_{\alpha/2} < t < t_{\alpha/2}$ then we accept $H_0$; otherwise we reject $H_0$

(b) If $t < t_{\alpha}$ then accept $H_0$, Otherwise reject $H_0$

(c) If $-t_{\alpha} < t$ then we accept $H_0$; otherwise we reject $H_0$.

Important Note :-

* Table values always given for one tail test

* Suppose we want to find table value of $t$ for 2 tail test corresponding to significance level $\alpha$, then we find table value for one tail test at $\frac{\alpha}{2}$ significance level.

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Problems :

(1) A.M-2018 Given a sample mean of 83, sample standard deviation of 12.5 and a sample size of 22, test the hypothesis that the value of population mean is 70 against the alternative that it is more than 70. Use the 0.025 significance level.

Soln

Given : $n = 22$ , $\mu = 70$ , $s = 12.5$ , $\bar{x} = 83$

$\alpha = 0.025 = 2.5\%$.

$H_0 : \mu = 70$

$H_1 : \mu > 70$ [ One tail test - Right ]

Degrees of freedom $= \nu = n-1$

$\nu = 22-1$

$\boxed{\nu = 21}$

<u>To find test statistic :</u>

$$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)} = \frac{83-70}{\left(\frac{12.5}{\sqrt{22-1}}\right)} = \frac{13}{\frac{12.5}{\sqrt{21}}}$$

$$= \frac{13 \times \sqrt{21}}{12.5} = 4.766$$

Now table value of $t_\alpha = 2.08$

Calculated $t$ value $>$ table value $t_\alpha$

$\therefore$ We reject $H_0$.

$\therefore$ Mean value of the population is greater than 70.

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(2) A machinist is making engine parts with axle diameters of 0.7 inch. A random sample of 10 parts shows a mean diameter of 0.742 inch with a standard deviation of 0.04 inch. Compute the statistic you would use to test, whether the work is meeting the specification.

AIM 2019

Soln

Given $n = 10$, $\bar{x} = 0.742$, $s = 0.04$, $\mu = 0.7$

$H_0 : \mu = 0.7$

$H_1 : \mu \neq 0.7$ [Two-tailed test]

Take $\alpha = 5\%$

Degrees of freedom $= \nu = n - 1$

$\nu = 10 - 1$

$\boxed{\nu = 9}$

__To find Test statistic:__

$$t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n-1}}\right)} = \frac{0.742 - 0.7}{\left(\frac{0.04}{\sqrt{10-1}}\right)} = \frac{0.742 - 0.7}{\left(\frac{0.04}{3}\right)}$$

$$= \frac{(0.742 - .7) \times 3}{(0.04)} = 3.15$$

Now table value of $t_\alpha$ corresponding to 5% (Two-tail) $\Big\}$ = table value of $t_\alpha$ corresponding to $5/2\ \%$ (One Tail)

= table of $t_\alpha$ corresponding to $2.5\%$

= table value of $t_\alpha$ corresponding to 0.025

$= 2.262$.

Conclusion:

If $-t_{\alpha/2} < t < t_{\alpha/2}$ then we accept $H_0$.

$-2.262 < 3.15 < 2.262$ is not true.

$\therefore$ We reject $H_0$. $\Rightarrow$ The product is not conforming the specification,

(3) A/M-2016 N/D-2017

A random sample of 10 boys had the following I.Q's : 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Do these data support the assumption of a population mean I.Q of 100? Find the reasonable range in which most of the mean I.Q values of samples of 10 boys lie.

Soln

Given $n=10$, $\mu=100$.

| | | | | | | | | | | | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$: | 70 | 120 | 110 | 101 | 88 | 83 | 95 | 98 | 107 | 100 | 972 |
| $x^2$: | $70^2$ | $120^2$ | $110^2$ | $101^2$ | $88^2$ | $83^2$ | $95^2$ | $98^2$ | $107^2$ | $100^2$ | 96312 |

$$\bar{x} = \frac{(\Sigma x)}{n} = \frac{972}{10} = 97.2$$

$$s^2 = \frac{\Sigma x^2}{n} - (\bar{x})^2 = \frac{96312}{10} - (97.2) = 183.36$$

$$s = \sqrt{183.36} = 13.5$$

$H_0$ : $\mu = 100$

$H_1$ : $\mu \neq 100$ (Two-tailed Test)

Take $\alpha = 5\% = 5/100 = 0.05$

Degrees of freedom $= \nu = n-1$

$$\boxed{\nu = 9}$$

To find Test statistic :-

$$t = \frac{\bar{x}-\mu}{\left(s/\sqrt{n-1}\right)} = \frac{97.2-100}{\left(\frac{13.5}{\sqrt{9}}\right)} = -0.62.$$

Table value $t_\alpha = 2.262$.

Here $-t_{\alpha/2} < t < t_{\alpha/2}$

i.e. $-2.262 < -0.62 < 2.262$.

So we accept $H_0$. i.e) These data support the assumption of population mean IQ of 100.

95 % confidence limits are given by

$$\bar{x} \pm 2.262 \frac{s}{\sqrt{n}}$$

$$= 97.2 \pm 2.262 \left(\frac{13.5}{\sqrt{10}}\right)$$

$$= 97.2 \pm 2.262 (4.269)$$

$$= 97.2 \pm 9.656$$

$$= 106.85 \text{ and } 87.54$$

$\therefore$ 95 % confidence limits within which the mean I.Q values of samples of 10 boys will lie is

$$\boxed{[87.54 , 106.85]}$$

(4) A/M 2017

A certain medicine administered to each of 10 patients resulted in the following increase in Blood pressure (B.P) 8, 8, 7, 5, 4, 1, 0, 0, −1, −1. Can it be concluded that the medicine was responsible for the increase in B.P 5% level of significance.

Soln :- Given $n = 10$.

| | | | | | | | | | | | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$: | 8 | 8 | 7 | 5 | 4 | 1 | 0 | 0 | −1 | −1 | 31 |
| $x^2$: | 64 | 64 | 49 | 25 | 16 | 1 | 0 | 0 | 1 | 1 | 221. |

$$\bar{x} = \frac{(\Sigma x)}{n} = \frac{31}{10} = 3.1$$

$$s^2 = \frac{(\Sigma x^2)}{n} - (\bar{x})^2 = \frac{221}{10} - (3.1)^2 = 22.1 - 9.61$$

$$s^2 = 12.49.$$

$$s = \sqrt{12.49} = 3.5$$

$H_0 : \mu = 0$ [No increase in B.P]

$H_1 : \mu \neq 0$ [Two tailed Test].

$\alpha = 5\% = 5/100 = .05 \Rightarrow \alpha/2 = 0.025$

Degrees of freedom $= \nu = n-1$

$\Rightarrow \boxed{\nu = 9}$

Test statistic $(t) = \dfrac{\bar{x} - \mu}{\left(s/\sqrt{n-1}\right)} = \dfrac{3.1 - 0}{3.5/\sqrt{9}} = \dfrac{3.1 \times 3}{3.5}$

$= 2.657$

Table value $t_{\alpha/2} = 2.262$.

If $-t_{\alpha/2} < t < t_{\alpha/2}$ we accept $H_0$.

Here $-2.262 < 2.657 < 2.262$ is not True.

∴ We reject $H_0$.

∴ There is increase in B.P.

# t - test For Difference of Means

Suppose we want to test, if two independent samples $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots y_{n_2}$ of sizes $n_1$ and $n_2$ have been drawn from two normal population with means $\mu_1$ and $\mu_2$ respectively.

Then we use t-test.

$H_0$ : Samples have been drawn from normal population with means $\mu_1$ and $\mu_2$ $(\mu_1 = \mu_2)$.

$H_1$ : Alternate hypothesis

Test statistic $t = \dfrac{(\bar{x} - \bar{y}) - (\mu_1 - \mu_2)}{S\sqrt{1/n_1 + 1/n_2}}$

Here $\bar{x} = \dfrac{(\Sigma x_i)}{n_1}$ , $\bar{y} = \dfrac{(\Sigma y_i)}{n_2}$

$$S^2 = \frac{1}{n_1+n_2-2}\left[\Sigma (x_i - \bar{x})^2 + \Sigma (y_i - \bar{y})^2\right]$$

Degrees of freedom $= n_1+n_2-2$

(or) $\nu = n_1+n_2-2$.

If $\mu_1 = \mu_2$

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad \text{where } S = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}$$

Conclusion :-

(Two tail) (a) If $-t_{\alpha/2} < t < t_{\alpha/2}$ , we accept $H_0$ ; otherwise reject $H_0$

(One Tail right) → (b) If $t < t_\alpha$ , we accept $H_0$ ; otherwise reject $H_0$

(one tail left) → (c) If $-t_\alpha < t$ we accept $H_0$ ; otherwise reject $H_0$.

Problems

(1) N/D 2009 Two horses A and B were tested according to the time (in seconds) to run a particular race with the following results.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Horse A : | 28 | 30 | 32 | 33 | 33 | 29 | 34 |
| Horse B : | 29 | 30 | 30 | 24 | 27 | 29 | — |

Test whether the horse A is running faster than B at 5 % level.

Soln Given $n_1 = 7$, $n_2 = 6$

$$\bar{x} = \frac{\Sigma x}{n_1} = \frac{28+30+32+33+33+29+34}{7} = 31.29$$

$$\bar{y} = \frac{\Sigma y}{n_2} = \frac{29+30+30+24+27+29}{6} = 28.17$$

$$s_1^2 = \frac{(\Sigma x^2)}{n_1} - (\bar{x})^2 = \frac{[28^2+30^2+32^2+33^2+33^2+29^2+34]}{7} - (31.29)^2$$

$$s_1^2 = 4.23$$

$$s_2^2 = \frac{\Sigma y^2}{n_2} - (\bar{y})^2 = \frac{29^2+30^2+30^2+24^2+27^2+29^2}{6} - (28.17)^2$$

$$s_2^2 = 4.28$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}$$

$$= \frac{7(4.23)+6(4.28)}{7+6-2}$$

$$S^2 = 5.03$$

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 > \mu_2$ [Horse A is faster than B]

$\alpha = 5\%$

This is one tail test.

Degrees of freedom $v = 7+6-2 = 11$

Test statistic

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{S^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$$

$$= \frac{31.29-28.17}{\sqrt{5.03\left(\frac{1}{7}+\frac{1}{6}\right)}} = 2.498$$

Table value $t_\alpha = 1.796$.

If $t < t_\alpha$ we accept

Here $t < t_\alpha$ is not True

(i) $2.498 < 1.796$ is not True.

∴ We reject $H_0$.

∴ A is running not faster than B

(2) A group of 10 rats fed on diet A and another group of 8 rats fed on diet B, recorded the following increase in weights (gms).

N/D 2011
A/M 2015

Diet A : 5, 6, 8, 1, 12, 4, 3, 9, 6, 10

Diet B : 2, 3, 6, 8, 10, 1, 2, 8.

Does it show superiority of diet A over diet B.

Soln:- Given $n_1 = 10$, $n_2 = 8$.

$$\bar{x} = \frac{\Sigma x}{n_1} = \frac{64}{10} = 6.4$$

$$\bar{y} = \frac{\Sigma y}{n_2} = \frac{40}{8} = 5$$

$$s_1^2 = \frac{\Sigma x^2}{n_1} - (\bar{x})^2$$

$$= \frac{512}{10} - (6.4)^2$$

$$s_1^2 = 10.24.$$

$$s_2^2 = \frac{\Sigma y^2}{n_2} - (\bar{y})^2$$

$$= \frac{282}{8} - (5)^2$$

$$= 10.25$$

Degrees of freedom $= n_1 + n_2 - 2$
$= 10 + 8 - 2$

$$\boxed{v = 16}$$

$\alpha = 5\% = 0.05$

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 > \mu_2$

(Diet A is superior than B)

One Tail Test – Right.

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{10(10.24) - 8(10.25)}{10 + 8 - 2}$$

$$S^2 = 11.525$$

Test statistic

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{6.4 - 5}{\sqrt{11.525\left(\frac{1}{10} + \frac{1}{8}\right)}}$$

$$= \frac{6.4 - 5}{\sqrt{11.525(0.1 + 0.125)}}$$

$$= \frac{1.4}{1.610318}$$

$$t = 0.8693$$

Table value $t_\alpha = 1.746$

If $t < t_\alpha$ we accept $H_0$.

Here $t < t_\alpha$ is ~~not~~ True.

$0.8693 < 1.746$

$\therefore$ We accept $H_0$.

$\therefore$ There is no significant difference b/w diets A & B.

③ AIM 2019 The independent samples from normal population with equal variance give the following:

| Sample | Size | Mean | S.D |
|---|---|---|---|
| 1 | 16 | 23.4 | 2.5 |
| 2 | 12 | 24.9 | 2.8 |

Is the difference b/w means significant?

Soln: S.D = Standard deviation

Given $n_1 = 16$ ; $\bar{x} = 23.4$ ; $s_1 = 2.5$ $\Rightarrow$ $s_1^2 = 6.25$

$n_2 = 12$ ; $\bar{y} = 24.9$ ; $s_2 = 2.8$ $\qquad$ $s_2^2 = 7.84$.

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{16(6.25) + 12(7.84)}{16 + 12 - 2} = 7.46$$

$H_0 : \mu_0 = \mu_2$ (There is no significant difference b/w means)

$H_1 : \mu_0 \neq \mu_2$ (Two tail test)

Take $\alpha = 5\% = 0.05$

$\Rightarrow \alpha/2 = 0.025$ Degrees of freedom $= v = n_1 + n_2 - 2 = 26$.

Test statistic
$$t = \frac{\bar{x} - \bar{y}}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{23.4 - 24.9}{\sqrt{7.46\left(\frac{1}{16} + \frac{1}{12}\right)}} = \frac{-1.5}{1.043032438}$$

$$t = -1.438$$

From table $t_{\alpha/2} = 2.048$.

If $-t_{\alpha/2} < t < t_{\alpha/2}$, we accept $H_0$.

$$-2.048 < -1.438 < 2.048.$$

$\therefore$ We accept $H_0$.

$\therefore$ There is no significant difference b/w means.

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(4) The nicotin content in milligram of 2 samples of Tobacco were found to be as follows.

A/M 2018

| | |
|---|---|
| Sample A | 24 27 26 21 25 |
| Sample B | 27 30 28 31 22 36 |

Can it be said that these samples were from normal population with the same mean? Test at 5% level of significance

Soln

Given $n_1 = 5$, $n_2 = 6$

$$\bar{x} = \frac{\Sigma x}{n_1} = \frac{24+27+26+21+25}{5} = 24.6$$

$$\bar{y} = \frac{\Sigma y}{n_2} = \frac{27+30+28+31+22+36}{6} = 29.$$

$$s_1^2 = \frac{(\Sigma x^2)}{n_1} - (\bar{x})^2 = \frac{24^2+27^2+26^2+21^2+25^2}{5} - (24.6)^2 = 4.24.$$

$$s_2^2 = \frac{(\Sigma y^2)}{n_2} - (\bar{y})^2 = \frac{27^2+30^2+28^2+31^2+22^2+36^2}{6} - (29)^2 = 18.$$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2} = \frac{5(4.24)+6(18)}{5+6-2} = 14.35$$

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$ [Two tail test]

$$\alpha = 5\% = \frac{5}{100} = 0.05$$

$$\alpha/2 = 0.025$$

Degrees of freedom $= n_1+n_2-2$

$$v = 6+5-2$$

$$\boxed{v=9}$$

Test statistic

$$t = \frac{\bar{x}-\bar{y}}{\sqrt{S^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{24.6-29}{\sqrt{14.35\left(\frac{1}{5}+\frac{1}{6}\right)}} = \frac{-4.4}{\sqrt{14.35(0.3666)}}$$

$$t = \frac{-4.4}{2.29383231}$$

$$t = -1.91818$$

From table $t_{\alpha/2} = 2.262.$

If $-t_{\alpha/2} < t < t_{\alpha/2}$ we accept $H_0$.

Here $-2.262 < -1.91818 < 2.262$.

$\therefore$ We accept $H_0$.

(c) Two samples are taken from normal populations with same mean.

## $\chi^2$ - test for Population Variance

$\chi^2$ may be read as "chi-square"

Let $x_1, x_2, \ldots, x_n$ be a random sample from a normal population with variance $\sigma^2$.

$H_0 : \sigma^2 = \sigma_0^2$

$H_1 : \sigma^2 \neq \sigma_0^2$

Degrees of freedom $= n-1$

(ie) $\boxed{\nu = n-1}$

Test statistic

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma_0}\right)^2 = \frac{ns^2}{\sigma_0^2}$$

where $s^2$ = variance of sample

Conclusion :-

If calculated $\chi^2$ < table $\chi^2$, then we accept $H_0$.

Otherwise reject $H_0$.

Problems

(1) A random sample of size 25 from a population gives the sample standard deviation 8.5. Test the hypothesis that the population standard deviation is 10 at 0.5 %

A/m-2018 (Reg-2008)

Soln Given $n = 25$, $s = 8.5$, $\sigma = 10$

$H_0 : \sigma^2 = \sigma_0^2$ [Population s.d is 10]

$H_1 : \sigma^2 \neq \sigma_0^2$

$\alpha = 0.5\,\% = 0.005$

Degrees of freedom $= n-1$

$\boxed{\nu = 24}$

Table $\chi^2$ value $= 45.558$

Here $\sigma_0 = 10$

Test statistic $\chi^2 = \frac{ns^2}{\sigma_0^2}$

$= \frac{25 \times (8.5)^2}{(10)^2} = 18.06.$

Calculated $\chi^2$ value < table $\chi^2$ value.

$\therefore$ we accept $H_0$.

(2) AIM-2017 It is beleived that the precision (as measured by the variance) of an instrument is nomore than 0.16. Write down the null and alternate hypothesis for testing this belief. Carry out the test at 1% level of significance given 11 measurements of the same subject on the instrument.

2.5 , 2.3 , 2.4 , 2.3 , 2.5 , 2.7 , 2.5 , 2.6 , 2.6 , 2.7 , 2.5

Soln:-

Given $\sigma^2 = 0.16$.

| $x$ | $x-\bar{x}$ $= x - 2.51$ | $(x-\bar{x})^2$ |
|---|---|---|
| 2.5 | −0.01 | 0.0001 |
| 2.3 | −0.21 | 0.0441 |
| 2.4 | −0.11 | 0.0121 |
| 2.3 | −0.21 | 0.0441 |
| 2.5 | −0.01 | 0.0001 |
| 2.7 | 0.19 | 0.0361 |
| 2.5 | −0.01 | 0.0001 |
| 2.6 | 0.09 | 0.0081 |
| 2.6 | 0.09 | 0.0081 |
| 2.7 | 0.19 | 0.0361 |
| 2.5 | −0.01 | 0.0001 |
| $\bar{x} = \frac{27.6}{11}$ $= 2.51$ | | $\Sigma(x-\bar{x})^2$ $= 0.1891$ |

$H_0 : \sigma^2 = \sigma_0^2$

$H_1 : \sigma^2 \neq \sigma_0^2$

$\alpha = 1\% = 0.01$

Degrees of freedom

$v = n-1 = 11-1$

$\boxed{v = 10}$

Table value $\psi^2 = 23.209$

Test statistic

$$\psi^2 = \frac{ns^2}{\sigma^2} = \frac{\Sigma(x-\bar{x})^2}{\sigma^2}$$

$$= \frac{0.1891}{0.16} = 1.182$$

Conclusion:-

If calculate $\psi^2$ value < table $\psi^2$ vale we accept $H_0$.

$1.182 < 23.209$.

$\therefore$ We accept $H_0$.

# Large Sample Test Based on Normal Distribution

## For Single Mean

Large sample :- A sample is large if sample size $n > 30$.

Set Ho : Null hypothesis : There is no significant difference

H1 : Alternate hypothesis : There is a significant difference

(OR) $H_0 : \mu =$ value specified

$H_1 : \mu \neq$ value specified

| Tests | Level of Significance ($\alpha$) | | |
|---|---|---|---|
| | 1% | 5% | 10% |
| Two-tailed | $Z_{\alpha/2} = 2.58$ | $Z_{\alpha/2} = 1.96$ | $Z_{\alpha/2} = 1.645$ |
| Right Tailed | $Z_{\alpha} = 2.33$ | $Z_{\alpha} = 1.645$ | $Z_{\alpha} = 1.28$ |
| Left Tailed | $-Z_{\alpha} = -2.33$ | $-Z_{\alpha} = -1.645$ | $-Z_{\alpha} = -1.28$ |

Test statistic

$$Z = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

where $\bar{x} \rightarrow$ mean of sample

$\mu \rightarrow$ mean of population

$\sigma \rightarrow$ population standard deviation

$n \rightarrow$ Sample size.

Conclusion :-

(i) For Two-tailed test, if $-Z_{\alpha/2} < Z < Z_{\alpha/2}$, we accept Ho.

(ii) For Right-tailed test, if $Z < Z_{\alpha}$, we accept Ho.

(iii) For Left-tailed test, if $-Z_{\alpha} < Z$, we accept Ho.

Otherwise reject Ho.

## Problems

(1) N/D-2016 A sample of 900 members has a mean 3.4 cm and standard deviation 2.61 cm. Is the sample from a large population of mean 3.25 cm and standard deviation of 2.61 cm? (Test at 5% level of significance)

Soln

Given $n=900$, $\bar{x}=3.4$, $s=2.61$, $\mu=3.25$, $\alpha=5\%$.

$H_0$ : There is no significant difference $[\mu=3.25]$

$H_1$ : There is a significant difference $[\mu\neq 3.25]$

Two-tailed test.

$\alpha=5\%$.

By table, $Z_{\alpha/2}=1.96$.

Test statistic $Z=\dfrac{\bar{x}-\mu}{\left(s/\sqrt{n}\right)}=\dfrac{3.4-3.25}{\left(\dfrac{2.61}{\sqrt{900}}\right)}$

$$=\frac{(3.4-3.25)\times\sqrt{900}}{2.61}$$

$$=1.724.$$

Conclusion :-

For two-tailed test,

If $-Z_{\alpha/2} < Z < Z_{\alpha/2}$, then we accept $H_0$.

Here $Z_{\alpha/2}=1.96$.

$\therefore -1.96 < 1.724 < 1.96$ which is True.

$\therefore$ We accept $H_0$.

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(2) The mean life time of a sample of 100 light bulbs produced by a company is computed to be 1570 hours with a standard deviation of 120 hours. If $\mu$ is the mean life time of all the bulbs produced by the company, test the hypothesis $\mu = 1600$ hours, against the alternative hypothesis $\mu \neq 1600$ hours with $\alpha = 0.05$ and $0.01$.

AIM2003

Soln

Given $n = 100$, $\bar{x} = 1570$, $s = 120$, $\mu = 1600$

We must check at $\alpha = 0.05$ & $\alpha = 0.01$

(i) $\alpha = 5\%$ & $\alpha = 1\%$.

$H_0 : \mu = 1600$

$H_1 : \mu \neq 1600$ [Two-tailed Test]

Test statistic $Z = \dfrac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)} = \dfrac{1570 - 1600}{\left(\frac{120}{\sqrt{100}}\right)}$

$$= \frac{-30 \times \sqrt{100}}{120}$$

$$= \frac{-30 \times 10}{120}$$

$$= -2.5$$

If $-Z_{\alpha/2} < Z < Z_{\alpha/2}$ we accept $H_0$.

| At $\alpha = 5\%$ | At $\alpha = 1\%$ |
|---|---|
| $Z_{\alpha/2} = 1.96$ | $Z_{\alpha/2} = 2.58$. |
| $-1.96 < -2.5 < 1.96$ is not True. | $-2.58 < -2.5 < 2.58$ is True. |
| $\therefore$ We reject $H_0$. | $\therefore$ We accept $H_0$. |

③ A normal population has a mean of 6.48 and standard deviation of 1.5. In a sample of 400 members mean is 6.75 and s.d is 1.5. Is the difference significant?

Soln: Given $n = 400$, $\bar{x} = 6.75$, $\mu = 6.48$, $s = 1.5$

$H_0$: There is no significant difference

$H_1$: There is a significant difference. [Two-tailed test]

Take $\alpha = 5\%$.

Test Statistic $$z = \frac{\bar{x} - \mu}{\left(s/\sqrt{n}\right)}$$

$$= \frac{6.75 - 6.48}{\left(\frac{1.5}{\sqrt{400}}\right)}$$

$$= 3.6$$

From table, $z_{\alpha/2} = 1.96$.

Conclusion :-

If $-z_{\alpha/2} < z < z_{\alpha/2}$ we accept $H_0$.

$$-1.96 < 3.6 \not< 1.96$$

$\therefore$ We reject $H_0$.

i.e) There is a significant difference.

(4) The mean breaking strength of the cables supplied by a manufacturer is 1800 with a standard deviation (S.D) of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. In order to test his claim, a sample of 50 cables is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1% level of significance?

Soln.

Given $\mu = 1800$, $s = 100$, $n = 50$, $\bar{x} = 1850$, $\alpha = 1\%$.

$H_0 : \mu = 1800$

$H_1 : \mu > 1800$. [Right Tailed Test]

Table value $Z_\alpha = 2.33$.

Test statistic $Z = \dfrac{\bar{x} - \mu}{\left(s/\sqrt{n}\right)}$

$$= \frac{1850 - 1800}{\left(100/\sqrt{50}\right)}$$

$$= \frac{50 \times \sqrt{50}}{100}$$

$$= 3.535$$

Conclusion :-

For Right-Tailed Test,

If $Z < Z_\alpha$, then we accept $H_0$.

$3.535 \not< 2.33$

$\therefore$ We reject $H_0$.

$\therefore$ We accept $H_1$.

$\therefore$ We may support the claim.

(5) The average number of defective articles per day in a certain factory is claimed to be less than the average of all the factories. The average of all the factories is 30.5. A random sample of 100 days showed the following distribution

| Class Limits : | 16-20 | 21-25 | 26-30 | 31-35 | 36-40 |
|---|---|---|---|---|---|
| No.of days : | 12 | 22 | 20 | 30 | 16 |

Is the average less than the figure for all the factories? Given $S = 6.35$, $\bar{x} = 28.8$, $\alpha = 1\%$.

Soln.

Given $n = 100$, $\mu = 30.5$, $s = 6.35$, $\bar{x} = 28.8$.

$H_0 : \mu = 30.5$

$H_1 : \mu < 30.5$ [Left Tailed Test]

$\alpha = 1\%$

Table $-Z_\alpha = -2.33$

Test Statistic $Z = \dfrac{\bar{x} - \mu}{\left(s/\sqrt{n}\right)} = \dfrac{28.8 - 30.5}{\left(6.35/\sqrt{100}\right)}$

$= -2.68.$

Conclusion :-

For Left tailed, If $-Z_\alpha < Z$ then we accept $H_0$.

$-2.33 < -2.68$ is not True.

$\therefore$ we reject $H_0$.

$\therefore$ We accept $H_1$.

## Large Sample test based on Normal distribution for difference of means :

$H_0 : \mu_1 = \mu_2$ (There is no significant difference between the sample means)

$H_1 : \mu_1 \neq \mu_2$ [Two tailed test]

$\mu_1 < \mu_2$ [Left tailed test]

$\mu_1 > \mu_2$ [Right tailed test]

Significance level $= \alpha$

Test statistic $$Z = \frac{(\bar{x} - \bar{y})}{\left(\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)}$$

(OR)

$$Z = \frac{(\bar{x} - \bar{y})}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

$\bar{x} \rightarrow$ First sample mean

$\bar{y} \rightarrow$ Second sample mean

$\sigma_1^2 \rightarrow$ First population variance

$\sigma_2^2 \rightarrow$ Second population Variance

$n_1 \rightarrow$ First Sample size

$n_2 \rightarrow$ Second sample size.

Conclusion :-

(a) For Two tailed test, If $-Z_{\alpha/2} < Z < Z_{\alpha/2}$, we accept $H_0$

(b) For Right Tailed test, If $Z < Z_{\alpha}$, we accept $H_0$

(c) For Left Tailed test, If $-Z_{\alpha} < Z$, we accept $H_0$.

Otherwise reject $H_0$.

## Problems

(1) N/M-2013 N/M 2017 The sales manager of a large company conducted a sample survey in states A and B taking 400 samples in each case. The results were in the following table. Test whether the average sales in the same in the 2 states at 1% level.

| | State A | State B |
|---|---|---|
| Average Sales | Rs. 2500 | Rs. 2200 |
| S.D | Rs : 400 | Rs : 550 |

Soln

Given $n_1 = 400$, $n_2 = 400$,

$\bar{x} = 2500$, $\bar{y} = 2200$

$S_1 = 400$; $S_2 = 550$.

$\alpha = 1\%$.

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$. [Two-tailed Test]

From table, $Z_{\alpha/2} = 2.58$.

Test statistic $Z = \dfrac{\bar{x} - \bar{y}}{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)} = \dfrac{2500 - 2200}{\left(\sqrt{\dfrac{(400)^2}{400} + \dfrac{(550)^2}{400}}\right.}$

$= \dfrac{300}{\sqrt{400 + 756.25}}$

$= \dfrac{300}{34.00367627}$

$= 8.82.$

Conclusion:-

If $-z_{\alpha/2} < z < z_{\alpha/2}$ we accept $H_0$.

$-2.58 < 8.82 < 2.58$ is not True.

∴ We reject $H_0$.

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(2) A sample of heights of 6400 Englishmen has a mean of 67.85 inches and a S.D of 2.56 inches, while a sample of heights of 1600 Australians has a mean of 68.55 inches and a S.D of 2.52 inches. Do the data indicate that Australians are on the avarage taller than Englishmen

N/D-2019

Soln Given

| Australians | Englishmen |
|---|---|
| $\bar{x} = 68.55$ | $\bar{y} = 67.85$ |
| $S_1 = 2.52$ | $S_2 = 2.56$ |
| $n_1 = 1600$ | $n_2 = 6400$ |

$$\Rightarrow \quad s_1^2 = (2.52)^2 = 6.3504$$
$$s_2^2 = (2.56)^2 = 6.5536.$$

Take $\alpha = 5\%$.

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 > \mu_2$ [Right tailed test]

From table, $Z_\alpha = 1.645$.

Test statistic $Z = \dfrac{\bar{x} - \bar{y}}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$

$$= \frac{68.55 - 67.85}{\sqrt{\frac{6.3504}{1600} + \frac{6.5536}{6400}}} = \frac{0.7}{0.070661163} = 9.9065$$

Conclusion:-

For Right tailed test, if $Z < Z_\alpha$ then we accept $H_0$

$9.9065 < 1.645$ is not True.

∴ We reject $H_0$.

∴ We accept $H_1$.

(i.e) Australians are on the average taller than Englishmen.

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(3) N/D-2017 A random sample of 100 bulbs from a company A shows a mean life 1300 hours and standard deviation of 82 hours. Another random sample of 100 bulbs from company B ~~showed~~ showed a mean life 1248 hours and standard deviation of 93 hours. Are the bulbs of company A superior to bulbs of company B at 5% level of significance?

Soln.

Given

| Company A | company B |
|---|---|
| $\bar{x} = 1300$ | $\bar{y} = 1248$ |
| $s_1 = 82$ | $s_2 = 93$ |
| $n_1 = 100$ | $n_2 = 100$ |

$$\alpha = 5\%$$

$$s_1^2 = (82)^2 = 6724$$

$$s_2^2 = 8649.$$

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 > \mu_2$ [A is superior than B]

[Right tailed test]

From table $Z_\alpha = 1.645.$

Test statistic $Z = \dfrac{\bar{x} - \bar{y}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$

$$= \frac{1300 - 1248}{\sqrt{\frac{6724}{100} + \frac{8649}{100}}} = \frac{52}{12.39879}$$

$$= 4.19$$

Conclusion :-

For Right tailed test, if $Z < Z_\alpha$ then we accept $H_0$.

Here $4.19 < 1.645$ is not true.

$\therefore$ We reject $H_0$.

$\therefore$ We accept $H_1$.

(i.e) Bulbs of company A are superior than that of B.

④ Given $\overline{x_1} = 72$ , $\overline{x_2} = 74$

N/D-2015

$s_1 = 8$ , $S_2 = 6$

$n_1 = 32$ , $n_2 = 36$

Test if the means are significant.

Soln.

Take $\overline{x_2} = \bar{y}$ , $\overline{x_1} = \bar{x}$

Given $\bar{x} = 72$ , $\bar{y} = 74$

$S_1 = 8$ , $S_2 = 6$   Take $\alpha = 5\%$

$n_1 = 32$ , $n_2 = 36$ .

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$. [Two tailed test]

From table, $Z_{\alpha/2} = 1.96$.

Test statistic $Z = \dfrac{\bar{x} - \bar{y}}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} = \dfrac{72 - 74}{\sqrt{8^2/32 + 6^2/36}}$

$= \dfrac{-2}{\sqrt{64/32 + 1/6}}$

$= \dfrac{-2}{1.471960144}$

$Z = -1.3587$

Conclusion:-

If $-Z_{\alpha/2} < Z < Z_{\alpha/2}$ we accept $H_0$

$-1.96 < -1.3587 < 1.96$ is True.

∴ We accept $H_0$.

∴ There is no significant different b/w means.

(5) A mathematics test was given to 50 girls and 75 boys. The girls made an average of 76 with an SD of 6 and the boys made an average of 82 with an SD of 2. Test whethere there is any difference between performance of boys and girls.

N/D-2015
A/M-2016

Soln :-

| Girls | Boys |
|---|---|
| $\bar{x} = 76$ | $\bar{y} = 82$ |
| $s_1 = 6$ | $s_2 = 2$ |
| $n_1 = 50$. | $n_2 = 75$ |

Take $\alpha = 5\%$.

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$ [Two tailed test]

From table, $Z_{\alpha/2} = 1.96$

Test statistic $Z = \dfrac{\bar{x} - \bar{y}}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} = \dfrac{76-82}{\sqrt{6^2/50 + 2^2/75}}$

$$= \frac{-6}{\sqrt{36/50 + 4/75}}$$

$$= \frac{-6}{0.87939373}$$

$$Z = -6.82$$

Conclusion :-

If $-Z_{\alpha/2} < Z < Z_{\alpha/2}$ then we accept $H_0$.

$-1.96 < -6.82 < 1.96$ is not True.

$\therefore$ We reject $H_0$.

$\therefore$ There is a significant different b/w performance of boys and girls.

(6) A/M-2017

The sales manager of a large company conducted a sample survey in states A and B taking 400 samples in each case. The results were

| | State A | State B |
|---|---|---|
| Average Sales | Rs 2500 | Rs 2200 |
| S.D | Rs 400 | Rs 550 |

Test whether the average sales is same in the same states at 1% level of significance?

Given $\bar{x} = 2500$, $\bar{y} = 2200$

$s_1 = 400$, $s_2 = 550$

$n_1 = 400$, $n_2 = 400$

set $H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$ [Two tailed]

$\alpha = 1\ \%$

From table $z_{\alpha/2} = 2.58$

Test statistic $$Z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{2500 - 2200}{\sqrt{\frac{(400)^2}{400} + \frac{(550)^2}{400}}}$$

$$= \frac{300}{\sqrt{400 + \frac{(550)^2}{400}}} = \frac{300}{34.00367627}$$

$$= 8.82$$

Conclusion:-

If $-z_{\alpha/2} < Z < z_{\alpha/2}$ we accept $H_0$.

$-2.58 < 8.82 < 2.58$ is not True.

$\therefore$ We reject $H_0$.

$\therefore$ The average sales are not same.

# Chi-Square Test for Goodness of fit

$\chi^2$-Test statistic of goodness of fit is given by

$$\chi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right],$$ where $O \rightarrow$ observed frequency
$E \rightarrow$ Expected frequency.

Degrees of freedom $= v = n-1$

Conclusion:-

If Calculated $\chi^2$ < table $\chi^2$, we accept $H_0$.

Otherwise reject $H_0$.

Note:

* By this test, we test whether differences between Observed and expected frequencies are significant or not.

Problems

① Five coins are tossed 320 times. The number of heads observed is given below

AIM-2018

| Number of Heads | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Frequency | 15 | 45 | 85 | 95 | 60 | 20 |

Examine whether the coins are unbiased. Use 5% level of significance

Soln

$H_0$ : The coins are unbiased

$H_1$ : The coins are biased.

Level of significance : $\alpha = 5\% = \frac{5}{100} = 0.05$

Degrees of freedom $= 6-1 = 5$.

$$\boxed{v = 5}$$

Table $\psi^2 = 11.070$

Test statistic : $\psi^2 = \sum \left[ \frac{(O-E)^2}{E} \right]$

$E$ = Total frequency $\times P(x_i)$.

where $P(x_i) = nC_x \, p^x q^{n-x}$, $x = 0,1,2,3,4,5$.

$p$ = probability of getting head $= \frac{1}{2}$

$q$ = probability of getting tail $= \frac{1}{2}$.

$P(0 \text{ head}) = 5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = 0.03$

$P(1 \text{ head}) = 5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 = 0.16$

$P(2 \text{ head}) = 5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = 0.31$

$P(3 \text{ head}) = 5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = 0.31$

$P(4 \text{ head}) = 5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) = 0.16$

$P(5 \text{ head}) = 5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = 0.03$

Given Total frequency = 320.

| No. of heads $x_i$ | O | $P(x_i)$ | $E = 320 \times P(x_i)$ | $\frac{(O-E)^2}{E}$ |
|---|---|---|---|---|
| 0 | 15 | 0.03 | 9.60 | 3.04 |
| 1 | 45 | 0.16 | 51.20 | 0.75 |
| 2 | 85 | 0.31 | 99.20 | 2.03 |
| 3 | 95 | 0.31 | 99.20 | 0.18 |
| 4 | 60 | 0.16 | 51.20 | 1.51 |
| 5 | 20 | 0.03 | 9.60 | 11.27 |

Total $= \sum \left( \frac{(O-E)^2}{E} \right) = 18.78$.

Conclusion :-

If Cal $\psi^2 <$ table $\psi^2$, we accept Ho.

$\psi^2 = \sum \left( \frac{(O-E)^2}{E} \right) = 18.78$

$18.78 < 11.070$ is not True. $\therefore$ We reject Ho $\Rightarrow$ The coins are biased.

(2) A Sample analysis of examination results of 1000 students were made and it was found that 260 failed, 110 first class, 420 second class and the rest obtained third class. Do these data support the general examination result in the ratio 2 : 1 : 4 : 3

AIM-2019

Soln

We seperate the students according to their results into 4 category.

$\therefore n = 4.$

$H_0$ : The results in four categories are in the ratio 2 : 1 : 4 : 3

$H_1$ : The results in four categories are not in the ratio 2 : 1 : 4 : 3

Take $\alpha = 5\% = 0.05$

Degrees of freedom $= \nu = n - 1$

$\boxed{\nu = 3}$

Table $\chi^2 = 7.815$.

Test statistic $\chi^2 = \sum \left[ \frac{(O-E)^2}{E} \right]$

The expected frequencies are $\frac{2}{10} \times 1000$, $\frac{1}{10} \times 1000$, $\frac{4}{10} \times 1000$, $\frac{3}{10} \times 1000$

They are 200, 100, 400, 300

| | O | E | O − E | $(O-E)^2$ | $\frac{(O-E)^2}{E}$ |
|---|---|---|---|---|---|
| Failures | 260 | 200 | 60 | 3600 | 18 |
| I | 110 | 100 | 10 | 100 | 1 |
| II | 420 | 400 | 20 | 400 | 1 |
| III | 210 | 300 | −90 | 8100 | 27 |

Conclusion :-

$\chi^2 = \sum \left[ \frac{(O-E)^2}{E} \right] = 47$

If Cal $\chi^2$ < table $\chi^2$, we accept $H_0$.

47 < 7.815 is not True.

$\therefore$ We reject $H_0$.

The results in four categories are not in the ratio 2 : 1 : 4 : 3

(3) NID-2010

The following table gives the number of aircraft accidents that occured during the various days of a week. Find whether the accidents are uniformly distributed over the week

| Days : | Sun | Mon | Tue | Wed | Thu | Fri | Sat |
|---|---|---|---|---|---|---|---|
| No of accidents : | 14 | 16 | 8 | 12 | 11 | 9 | 14 |

Soln :- Given $n = 7$

$H_0$ : The accidents are uniformly distributed

$H_1$ : The accidents are not uniformly distributed.

$\alpha = 5\% = 0.05$

Degrees of freedom $v = n - 1$

$\boxed{v = 6}$

Table $\chi^2 = 12.592$

Total number of accidents = 84

Expected number of accidents $= \frac{84}{7} = 12.$

| | O | E | (O−E) | $(O-E)^2$ | $\frac{(O-E)^2}{E}$ |
|---|---|---|---|---|---|
| Sun | 14 | 12 | 2 | 4 | 0.333 |
| Mon | 16 | 12 | 4 | 16 | 1.333 |
| Tue | 8 | 12 | −4 | 16 | 1.333 |
| Wed | 12 | 12 | 0 | 0 | 0 |
| Thu | 11 | 12 | −1 | 1 | 0.083 |
| Fri | 9 | 12 | −3 | 9 | 0.75 |
| Sat | 14 | 12 | 2 | 4 | 0.333 |

$$\chi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right] = 4.165.$$

Conclusion :

If cal $\chi^2$ < table $\chi^2$, we accept $H_0$.

$4.165 < 12.592$ is True.

$\therefore$ We accept $H_0$.

$\therefore$ The accidents are uniformly distributed.

## $\chi^2$ - test to test the independence of Attributes

**Type (1)** Given $2 \times 2$ contigency table

| | |
|---|---|
| a | b |
| c | d |

Test statistic $\chi^2 = \dfrac{(ad-bc)^2 [a+b+c+d]}{(a+c)(b+d)(a+b)(c+d)}$

Degrees of freedom $= 1$

Conclusion :-

If cal $\chi^2 <$ table $\chi^2$, we accept $H_0$.

Otherwise reject $H_0$.

**Type (2) :** Given other than $2 \times 2$ table.

Test statistic $\chi^2 = \sum \left[ \dfrac{(O-E)^2}{E} \right]$

$E \rightarrow$ Expected frequency.

$E = \dfrac{\text{corresponding row total} \times \text{corresponding column total}}{\text{Grand Total}}$

Degrees of freedom $= [\text{No. of rows} - 1] \times [\text{No. of columns} - 1]$

Conclusion :-

If cal $\chi^2 <$ table $\chi^2$, we accept $H_0$

Otherwise reject $H_0$.

Note

* Coefficient of attributes $= \dfrac{ad-bc}{ad+bc}$
[For $2 \times 2$ contigency table]

Problems

(1) Find if there is any association between extravagance in fathers and extravagance in sons from the following data

M/J 2013

| | Extravagant Father | Misely Father |
|---|---|---|
| Extravagant Son | 327 | 741 |
| Misely Son | 545 | 234 |

Determine the coefficient of association also.

Given $2 \times 2$ table. $a = 327$, $b = 741$, $c = 545$, $d = 234$

Soln. Ho: Extravagance in sons and Fathers are not significant

[There is no significant difference b/w extravagancy of fathers & Son]

H1: Significant.

$\alpha = 5\% = 0.05$

Degrees of freedom $= 1$. $\Rightarrow \boxed{v = 1}$

Table $\chi^2$ value $= 3.841$

Test statistic $\chi^2 = \dfrac{(ad - bc)\,[a+b+c+d]}{(a+b)(c+d)\,(a+c)(b+d)}$

$$= \frac{[(327)(234) - (545)(741)]^2 \times [327 + 741 + 545 + 234]}{(327 + 741)(545 + 234)(327 + 545)(741 + 234)}$$

$= 230.24$.

Conclusion:- If cal $\chi^2$ < table $\chi^2$ we accept Ho.

$230.24 < 3.841$ is not true.

So we reject Ho.

Coefficient of attributes $= \dfrac{ad - bc}{ad + bc} = \dfrac{(327)(234) - (741)(545)}{(327)(234) + (741)(545)}$

$= -0.6814$

M/J-2013

(2) 1000 students at college level were graded according to their I.Q and their economic conditions. What conclusion can you draw from the following data.

| Economic conditions | I.Q Level | |
|---|---|---|
| | High | Low |
| Rich | 460 | 140 |
| Poor | 240 | 160 |

Soln Given 2 x 2 table

$a = 460$ , $b = 140$
$c = 240$ , $d = 160$ .

$H_0$ : There is no significant difference b/w economic condition and I.Q level

$H_1$ : There is significant difference.

$\alpha = 5\% = 0.05$

Degrees of freedom = 1.

$\boxed{V=1}$

From table, $\chi^2$ table = 3.841 .

Test statistic $\chi^2 = \dfrac{(ad-bc)^2[a+b+c+d]}{(a+b)(c+d)(a+c)(b+d)}$

$= \dfrac{[(460)(160)-(140)(240)]^2[460+140+240+160]}{(460+140)(240+160)(460+240)(140+160)}$

$= 31.7460$ .

Conclusion

If cal $\chi^2 <$ table $\chi^2$ we accept Ho

$31.7460 < 3.841$ is not True.

So we reject Ho.

There is significant difference b/w attributes.

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(3) A/M-2018

Mechanical engineers testing a new arc welding technique, classified welds both with respect to appearance and an X-ray inspection.

| X-ray \ Appearance | Bad | Normal | Good | Total |
|---|---|---|---|---|
| Bad | 20 | 7 | 3 | 30 |
| Normal | 13 | 51 | 16 | 80 |
| Good | 7 | 12 | 21 | 40 |
| Total | 40 | 70 | 40 | 150 |

Test for independence using 0.05 level of significance.

Soln:-

$H_0$ : They are independent

$H_1$ : They are dependent.

$\alpha = 0.05$ , degrees of freedom $= [\text{No. of rows} - 1] \times [\text{No. of columns} - 1]$

$= 2 \times 2 = 4.$

Table $\chi^2 = 9.488$

Test statistic $\chi^2 = \sum \left[ \frac{(O-E)^2}{E} \right]$

$$E = \frac{\text{corresponding row total} \times \text{corresponding column total}}{\text{Grand total}}.$$

| O | E | $(O-E)^2$ | $\frac{(O-E)^2}{E}$ |
|---|---|---|---|
| 20 | $\frac{30 \times 40}{150} = 8$ | 144 | 18 |
| 7 | $\frac{30 \times 70}{150} = 14$ | 49 | 3.5 |
| 3 | $\frac{30 \times 40}{150} = 8$ | 25 | 3.13 |
| 13 | $\frac{80 \times 40}{150} = 21.33$ | 69.39 | 3.25 |
| 51 | $\frac{80 \times 70}{150} = 37.33$ | 186.87 | 5.01 |
| 16 | $\frac{80 \times 40}{150} = 21.33$ | 28.41 | 1.33 |
| 7 | $\frac{40 \times 40}{150} = 10.67$ | 13.47 | 1.26 |
| 12 | $\frac{40 \times 70}{150} = 18.67$ | 44.49 | 2.38 |
| 21 | $\frac{40 \times 40}{150} = 10.67$ | 106.71 | 10 |

Total $\chi^2 = \sum \left( \frac{(O-E)^2}{E} \right) = 47.86.$

Conclusion: If cal $\chi^2 <$ table $\chi^2$ we accept $H_0$.

$47.86 < 9.488$ is not True. So we reject $H_0$.

$\Rightarrow$ X ray and appearance are dependent.

# F-test [Test for Variance]

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

$\alpha = 5\%$ or $\alpha = 1\%$

Given 2 samples.

We want to test if there is any significant different between given two variances. ~~of samples~~.

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} , \quad S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}.$$

Here $n_1$ = 1st Sample size, $n_2$ = 2nd sample size

$s_1$ = 1st Sample variance, $s_2$ = 2nd sample variance.

$$s_1^2 = \frac{\Sigma(x^2)}{n_1} - (\bar{x})^2 , \quad s_2^2 = \frac{\Sigma(y)^2}{n_2} - (\bar{y})^2$$

Test statistic

If $S_1^2 > S_2^2$ then $F = \frac{S_1^2}{S_2^2}$ and $v_1 = n_1 - 1$, $v_2 = n_2 - 1$

If $S_2^2 > S_1^2$ then $F = \frac{S_2^2}{S_1^2}$ and $v_1 = n_2 - 1$, $v_2 = n_1 - 1$

Next calculate the table F value at '$\alpha$' level of significance with $v_1, v_2$ degrees of freedom.

Conclusion

If cal F value < table F value then we accept $H_0$.

Otherwise reject $H_0$.

Note

$$S_1^2 = \frac{\Sigma(x - \bar{x})^2}{n_1 - 1} = \frac{\text{Sum of squares of deviation from mean}}{n_1 - 1}$$

$$S_2^2 = \frac{\Sigma(y - \bar{y})^2}{n_2 - 1} = \frac{\text{Sum of Squares of deviation from mean}}{n_2 - 1}.$$

① A/M-2015 N/D-2013

A group of 10 rats fed on diet A and another group of 8 rats fed on diet B, recorded the following increase in weight.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Diet A : | 5 | 6 | 8 | 1 | 12 | 4 | 3 | 9 | 6 | 10 |
| Diet B : | 2 | 3 | 6 | 8 | 10 | 1 | 2 | 8 | | |

Find if the variances are significantly different.

Soln Given $n_1 = 10$, $n_2 = 8$.

$$\bar{x} = \frac{\Sigma x}{n_1} = \frac{5+6+8+1+12+4+3+9+6+10}{10} = 6.4$$

$$\bar{y} = \frac{\Sigma y}{n_2} = \frac{2+3+6+8+10+1+2+8}{8} = 5$$

$$s_1^2 = \frac{\Sigma x^2}{n_1} - (\bar{x})^2 = \frac{512}{10} - (6.4)^2 = 10.24$$

$$s_2^2 = \frac{\Sigma y^2}{n_2} - (\bar{y})^2 = \frac{282}{8} - 25 = 10.25.$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{10 \times 10.24}{10-1} = 11.3777$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{8 \times 10.25}{7} = 11.7143.$$

Here $S_2^2 > S_1^2$.

$\therefore v_1 = n_2 - 1 = 8-1$ and $v_2 = n_1 - 1 = 10-1$

$\boxed{v_1 = 7}$ $\boxed{v_2 = 9}$

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

$\alpha = 0.05$.

Table value of $F = 3.29$.

Test Statistic $F = \frac{S_2^2}{S_1^2} = \frac{11.7143}{11.3777} = 1.02958.$

Conclusion :-

If cal F < table F, we accept $H_0$. Otherwise reject $H_0$.

$1.02958 < 3.29$ is True.

So we accept $H_0$.

(2) N/D - 2016

Two random samples give the following results.

| Sample | Size | Sample mean | Sum of squares of deviation from mean |
|---|---|---|---|
| I | 10 | 15 | 90 |
| II | 12 | 14 | 108 |

Test whether the samples could have come from the same normal population.

Soln: Given $n_1 = 10$, $\bar{x} = 15$, $\Sigma(x-\bar{x})^2 = 90$

$n_2 = 12$, $\bar{y} = 14$, $\Sigma(y-\bar{y})^2 = 108$.

$$S_1^2 = \frac{\Sigma(x-\bar{x})^2}{n_1 - 1} = \frac{90}{9} = 10$$

$$S_2^2 = \frac{\Sigma(y-\bar{y})^2}{n_2 - 1} = \frac{108}{11} = 9.8181.$$

Here $S_1^2 > S_2^2$.

$\therefore$ $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$

$\boxed{v_1 = 9}$ $\boxed{v_2 = 11}$

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

$\alpha = 0.05$, $v_1 = 9$, $v_2 = 11$.

Table $F = 2.90$

Test statistic $F = \dfrac{S_1^2}{S_2^2} = \dfrac{10}{9.8181} = 1.019$

Conclusion :-

If cal F < table F, we accept $H_0$.

Other wise reject $H_0$.

$1.019 < 2.90$ is True

So we accept $H_0$.

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③

M/J 2014

A/M 2017

Two independent samples of sizes 9 and 7 from a normal population had the following values of the variables.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Sample I: | 18 | 13 | 12 | 15 | 12 | 14 | 16 | 14 | 15 |
| Sample II: | 16 | 19 | 13 | 16 | 18 | 13 | 15 | | |

Do the estimates of the population variances differ significally at 5% level ?

Soln :-

Given $n_1 = 9$ , $n_2 = 7$.

$$\bar{x} = \frac{\Sigma x}{n_1} = \frac{18+13+12+15+12+14+16+14+15}{9} = \frac{129}{9} = 14.3333$$

$$\bar{y} = \frac{\Sigma y}{n_2} = \frac{16+19+13+16+18+13+15}{7} = \frac{110}{7} = 15.7143$$

$$s_1^2 = \frac{\Sigma(x^2)}{n_1} - (\bar{x})^2 = \frac{1879}{9} - (14.3333)^2 = 3.3342$$

$$s_2^2 = \frac{\Sigma(y)^2}{n_2} - (\bar{y})^2 = \frac{1760}{7} - (15.7143)^2 = 4.4894$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{9 \times 3.3342}{8} = 3.751$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{7 \times 4.4894}{6} = 5.2376$$

Here $S_2^2 > S_1^2$

$\therefore$ $v_1 = n_2 - 1 = 6$

$v_2 = n_1 - 1 = 8$

$H_0 : \sigma_1^2 = \sigma_2^2$

$H_1 : \sigma_1^2 \neq \sigma_2^2$

$\alpha = 5\% = 0.05$ , $v_1 = 6$ , $v_2 = 8$

Table $F = 3.58$

Test Statistic $F = \frac{S_2^2}{S_1^2} = \frac{5.2376}{3.751} = 1.3963$

Conclusion:-

If cal F < table F, we accept $H_0$.

$1.3963 < 3.58$ is True.

So we accept $H_0$.

$\Rightarrow$ There is no significant different b/w variances.

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## One way Classification

(1) A/m-2004 There are three main brands of a certain powder. A set of 120 sample values is examined and found to be allocated among four groups (A, B, C and D) and three brands (I, II, III) as shown here under :

| Brands | Groups | | | |
|---|---|---|---|---|
| | A | B | C | D |
| I | 0 | 4 | 8 | 15 |
| II | 5 | 8 | 13 | 6 |
| III | 8 | 19 | 11 | 13 |

Is there any significant difference in brands preference? Answer at 5% level.

Soln

Ho : There is no significant difference between brands

H1 : There is a significant difference between brands.

| Brands | Groups | | | | Total | $X_1^2$ | $X_2^2$ | $X_3^2$ | $X_4^2$ |
|---|---|---|---|---|---|---|---|---|---|
| | A $(X_1)$ | B $(X_2)$ | C $(X_3)$ | D $(X_4)$ | | | | | |
| I $(Y_1)$ | 0 | 4 | 8 | 15 | $\Sigma Y_1 = 27$ | 0 | 16 | 64 | 225 |
| II $(Y_2)$ | 5 | 8 | 13 | 6 | $\Sigma Y_2 = 32$ | 25 | 64 | 169 | 36 |
| III $(Y_3)$ | 8 | 19 | 11 | 13 | $\Sigma Y_3 = 51$ | 64 | 361 | 121 | 169 |
| | $\Sigma X_1 = 13$ | $\Sigma X_2 = 31$ | $\Sigma X_3 = 32$ | $\Sigma X_4 = 34$ | $T = 110$ | $\Sigma X_1^2 = 89$ | $\Sigma X_2^2 = 441$ | $\Sigma X_3^2 = 354$ | $\Sigma X_4^2 = 430$ |

Step(1) $N = 12$

Step(2) $T = 110$

Step(3) $T^2/N = \frac{(110)^2}{12} = 1008.33$

Step(4)

$$TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$$

$$= 89 + 441 + 354 + 430 - 1008.33$$

$$= 305.67.$$

Step(5)

$$SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} - \frac{T^2}{N}$$ (Here $N_2$ = no of elts in each row)

$$= \frac{(27)^2}{4} + \frac{(32)^2}{4} + \frac{(51)^2}{4} - 1008.33$$

$$= 182.25 + 256 + 650.25 - 1008.33$$

$$= 80.17$$

Step (6)

$$SSE = TSS - SSR = 305.67 - 80.17 = 225.5$$

Step(7) ANOVA TABLE.

| Source of Variation | SS | d.f | MS | Variance ratio | Table value at 5% |
|---|---|---|---|---|---|
| Between rows | SSR $= 80.17$ | $r-1$ $= 3-1$ $= 2$ | $MSR = \frac{SSR}{r-1}$ $= \frac{80.17}{2}$ $= 40.085$ | $F_R = \frac{MSR}{MSE}$ $= \frac{40.08}{20.06} = 1.99$ | $F_R(2,9)$ $= 4.26$ |
| Error | SSE $= 225.5$ | $N-r$ $= 12-3$ $= 9$ | $MSE = \frac{SSE}{N-r}$ $= \frac{225.5}{9}$ $= 20.06$ | | |
| Total | 305.67 | | | | |

Conclusion:-

If cal $F_R$ < table $F_R$, we accept $H_0$.

$1.99 < 4.26$ is True.

So we accept $H_0$.

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(2) A completely randomised design experiment with 10 plots (2)

N/D-2007 and 3 treatments gave the following results:

| Plot No : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Treatment : | A | B | C | A | C | C | A | B | A | B |
| Yield : | 5 | 4 | 3 | 7 | 5 | 1 | 3 | 4 | 1 | 7 |

Analyse the results for treatment effects.

Soln:-

| A | B | C |
|---|---|---|
| 5 | 4 | 3 |
| 7 | 4 | 5 |
| 3 | 7 | 1 |
| 1 | – | – |

$H_0$ : There is no significant difference in treatments

$H_1$ : There is a significant difference in treatments.

| $X_1$ A | $X_2$ B | $X_3$ C | Total | $X_1^2$ | $X_2^2$ | $X_3^2$ |
|---|---|---|---|---|---|---|
| 5 | 4 | 3 | 12 | 25 | 16 | 9 |
| 7 | 4 | 5 | 16 | 49 | 16 | 25 |
| 3 | 7 | 1 | 11 | 9 | 49 | 1 |
| 1 | – | – | 1 | 1 | – | – |
| $\Sigma X_1 = 16$ | $\Sigma X_2 = 15$ | $\Sigma X_3 = 9$ | $T = 40$ | $\Sigma X_1^2 = 84$ | $\Sigma X_2^2 = 81$ | $\Sigma X_3^2 = 35$ |

Step(1) $N = 10$

Step(2) $T = 40$

Step(3) $T^2/N = \frac{1600}{10} = 160$

Step(4) $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 - T^2/N$

$= 84 + 81 + 35 - 160$

$= 40$

Step(5)

$$SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} - \frac{T^2}{N}.$$

Here $N_1 \rightarrow$ no. of elements in each column

$$= \frac{(16)^2}{4} + \frac{(15)^2}{3} + \frac{(9)^2}{3} - 160$$

$$= 64 + 75 + 27 - 160$$

$$SSC = 6$$

Step(6)

$$SSE = TSS - SSC$$

$$= 40 - 6 = 34.$$

Step(7) ANOVA TABLE.

| Source of Variation | Sum of Squares (SS) | d.f | M.S | Variance ratio | Table value at 5% |
|---|---|---|---|---|---|
| Between columns | $SSC = 6$ | $C-1$ $= 3-1$ $= 2$ | $MSC = \frac{SSC}{C-1} = \frac{6}{2}$ $= 3$ | $MSE > MSC$ $\therefore F_c = \frac{MSE}{MSC}$ | $\alpha = 0.05$ $v_1 = 7$ $v_2 = 2$ |
| Error | $SSE = 34$ | $N-C$ $= 10-3$ $= 7$ | $MSE = \frac{SSE}{N-C} = \frac{34}{7}$ $= 4.86$ | $= \frac{4.86}{3}$ $= 1.62$ | Table $F_c$ $= 19.35$ |

Conclusion :-

If cal $F_c$ < table $F_c$ we accept $H_0$.

$1.62 < 19.35$ is True

So we accept $H_0$.

(3) The following table shows the lives in hours of four brands of electric lamps.

A/M-2008
N/D-2011
M/J-2011
A/M-2015

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Brand A: | 1610 | 1610 | 1650 | 1680 | 1700 | 1720 | 1800 | – |
| B: | 1580 | 1640 | 1640 | 1700 | 1750 | – | – | – |
| C: | 1460 | 1550 | 1600 | 1620 | 1640 | 1660 | 1740 | 1820 |
| D: | 1510 | 1520 | 1530 | 1570 | 1600 | 1680 | – | – |

Perform an analysis of variance test the homogneity of the mean lives of the four brands of Lamps.

Soln :-

$H_0$ : There is no significant difference between 4 brands.

$H_1$ : There is significant difference between 4 brands.

Arranging A, B, C, D into columns.
(Subtract 1600 then divide by 10)

| $x_1$ (A) | $x_2$ (B) | $x_3$ (C) | $x_4$ (D) | Total | $x_1^2$ | $x_2^2$ | $x_3^2$ | $x_4^2$ |
|---|---|---|---|---|---|---|---|---|
| 1 | −2 | −14 | −9 | −24 | 1 | 4 | 196 | 81 |
| 1 | 4 | −5 | −8 | −8 | 1 | 16 | 25 | 64 |
| 5 | 4 | 0 | −7 | 2 | 25 | 16 | 0 | 49 |
| 8 | 10 | 2 | −3 | 17 | 64 | 100 | 4 | 9 |
| 10 | 15 | 4 | 0 | 29 | 100 | 225 | 16 | 0 |
| 12 | – | 6 | 8 | 26 | 144 | – | 36 | 64 |
| 20 | – | 14 | – | 34 | 400 | – | 196 | – |
| – | – | 22 | – | 22 | – | – | 484 | – |
| $\Sigma x_1 = 57$ | $\Sigma x_2 = 31$ | $\Sigma x_3 = 29$ | $\Sigma x_4 = -19$ | $T = 98$ | $\Sigma x_1^2 = 735$ | $\Sigma x_2^2 = 361$ | $\Sigma x_3^2 = 957$ | $\Sigma x_4^2 = 267$ |

Step(1) $N = 26$

Step(2) $T = 98$

Step(3) $\frac{T^2}{N} = \frac{(98)^2}{26} = 369.39$

Step(4)

$$TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - T^2/N$$

$$= 735 + 361 + 957 + 267 - 369.39$$

$$= 1950.61$$

Step(5)

$$SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_P} + \frac{(\Sigma x_4)^2}{N_4} - \frac{T^2}{N}$$

$$= \frac{(57)^2}{7} + \frac{(31)^2}{5} + \frac{(29)^2}{8} + \frac{(-19)^2}{6} - 369.39$$

$$= \frac{3249}{7} + \frac{961}{5} + \frac{841}{8} + \frac{361}{6} - 369.39$$

$$= 464.14 + 192.2 + 105.13 + 60.17 - 369.39$$

$$= 452.25$$

Step(6)

$$SSE = TSS - SSC = 1950.61 - 452.25 = 1498.36$$

Step(7) ANOVA TABLE

| Source of Variation | SS | d.f | MS | Variance ratio | Table value at 5 % |
|---|---|---|---|---|---|
| Between columns | $SSC = 452.25$ | $C-1 = 3$ | $MSC = \frac{SSC}{C-1} = \frac{452.25}{3} = 150.75$ | $MSC > MSE$<br>$F_c = \frac{MSC}{MSE}$ | $\alpha = 0.05$<br>$V_1 = 3$<br>$V_2 = 22$ |
| Error | $SSE = 1498.36$ | $N-C = 26-4 = 22$ | $MSE = \frac{1498.36}{22} = 68.11$ | $= \frac{150.75}{68.11} = 2.21$ | Table $F_c = 3.05$ |

Conclusion:

If cal $F_c <$ table $F_c$ we accept $H_0$.

$2.21 < 3.05$.

$\therefore$ We accept $H_0$.

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## Two-way Classification

(1) AIM-2011 NID-2015

An experiment was designed to study the performance of 4 different detergents for cleaning fuel injectors. The following "cleanliness" readings were obtained with specially designed equipment for 12 tanks of gas distributed over 3 different models of engines.

| | Engine 1 | Engine 2 | Engine 3 | Total |
|---|---|---|---|---|
| Detergent A | 45 | 43 | 51 | 139 |
| Detergent B | 47 | 46 | 52 | 145 |
| Detergent C | 48 | 50 | 55 | 153 |
| Detergent D | 42 | 37 | 49 | 128 |
| Total | 182 | 176 | 207 | 565 |

Perform the ANOVA and test at 0.01 level of significance, whether there are differences in the detergents or in the engines.

Soln:- Subtract 50 from each data

| Detergent | Engine 1 ($X_1$) | Engine 2 ($X_2$) | Engine 3 ($X_3$) | Total | $X_1^2$ | $X_2^2$ | $X_3^2$ |
|---|---|---|---|---|---|---|---|
| A ($Y_1$) | $-5$ | $-7$ | 1 | $\Sigma Y_1 = -11$ | 25 | 49 | 1 |
| B ($Y_2$) | $-3$ | $-4$ | 2 | $\Sigma Y_2 = -5$ | 9 | 16 | 4 |
| C ($Y_3$) | $-2$ | 0 | 5 | $\Sigma Y_3 = 3$ | 4 | 0 | 25 |
| D ($Y_4$) | $-8$ | $-13$ | $-1$ | $\Sigma Y_4 = -22$ | 64 | 169 | 1 |
| | $\Sigma X_1 = -18$ | $\Sigma X_2 = -24$ | $\Sigma X_3 = 7$ | $T = -35$ | $\Sigma X_1^2 = 102$ | $\Sigma X_2^2 = 234$ | $\Sigma X_3^2 = 31$ |

$H_0$ : There is no significant difference between given column treatments & row treatments.

$H_1$ : There is a significant difference between given column treatments & row treatments.

Step(1) $N = 12$

Step(2) $T = -35$

Step(3) $T^2/N = \frac{(-35)^2}{12} = 102.08$

**Step(4)** $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 - \frac{T^2}{N}$

$= 102 + 234 + 31 - 102.08$

$TSS = 264.92$

**Step(5)** $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(-18)^2}{4} + \frac{(-24)^2}{4} + \frac{(7)^2}{4} - 102.08$

$= 81 + 144 + 12.25 - 102.08$

$SSC = 135.17$

**Step(6)** $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(-11)^2}{3} + \frac{(-5)^2}{3} + \frac{(3)^2}{3} + \frac{(-22)^2}{3} - 102.08$

$= 110.91$

**Step(7)**

$SSE = TSS - SSC - SSR = 264.92 - 135.17 - 110.91$

$SSE = 18.84.$

| Source of Variation | SS | d.f | MS | Variance ratio | Table value $\alpha = 1\% = 0.01$ |
|---|---|---|---|---|---|
| Between columns | $SSC = 135.17$ | $c-1 = 3-1 = 2$ | $MSC = \frac{SSC}{c-1} = \frac{135.17}{2} = 67.585$ | $MSC > MSE$ $F_C = \frac{MSC}{MSE} = \frac{67.585}{3.14} = 21.52$ | $v_1 = 2, v_2 = 6$ Table $F_C = 10.92$ |
| Between rows | $SSR = 110.91$ | $r-1 = 4-1 = 3$ | $MSR = \frac{SSR}{r-1} = \frac{110.91}{3} = 36.97$ | $MSR > MSE$ $F_R = \frac{MSR}{MSE} = \frac{36.97}{3.14} = 11.77$ | $v_1 = 3, v_2 = 6$ Table $F_R = 9.78$ |
| Error | $SSE = 18.84$ | $N-c-r+1 = 12-3-4+1 = 6$ | $MSE = \frac{SSE}{N-c-r+1} = \frac{18.84}{6} = 3.14$ | | |
| | | | | | |

**Conclusion**

(i) Cal $F_C <$ Table $F_C$ we accept $H_0$.

$21.52 < 10.92$ is not True. So we reject $H_0$.

(ii) Cal $F_R <$ Table $F_R$ we accept $H_0$.

$11.77 < 9.78$ is not True. So we reject $H_0$.

(2) AIM-2017 AIM-2010

A set of data involving four tropical feed stuffs A, B, C, D tried on 20 chicks is given below. All the twenty chicks are treated alike in all respects except the feeding treatments and each feeding treatment is given to 5 chicks. Analyse the data.

Weight gain on baby chicks fed on different feeding materials composed of tropical feed stuffs.

| | | | | | | Total |
|---|---|---|---|---|---|---|
| A | 55 | 49 | 42 | 21 | 52 | 219 |
| B | 61 | 112 | 30 | 89 | 63 | 355 |
| C | 42 | 97 | 81 | 95 | 92 | 407 |
| D | 169 | 137 | 169 | 85 | 154 | 714 |
| | | | | | Grand Total | 1695 |

Soln

Ho : There is no significant difference b/w column treatments & row treatments.

H1 : There is a significant difference b/w column treatments & row treatments

Subtract 50 from each value.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Total | $x_1^2$ | $x_2^2$ | $x_3^2$ | $x_4^2$ | $x_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A ($Y_1$) | 5 | −1 | −8 | −29 | 2 | $\Sigma Y_1 =$ −31 | 25 | 1 | 64 | 841 | 4 |
| B ($Y_2$) | 11 | 62 | −20 | 39 | 13 | $\Sigma Y_2 =$ 105 | 121 | 3844 | 400 | 1521 | 169 |
| C ($Y_3$) | −8 | 47 | 31 | 45 | 42 | 157 $\Sigma Y_3 =$ | 64 | 2209 | 961 | 2025 | 1764 |
| D ($Y_4$) | 119 | 87 | 119 | 35 | 104 | 464 $\Sigma Y_4 =$ | 14161 | 7569 | 14161 | 1225 | 10816 |
| | $\Sigma x_1 =$ 127 | $\Sigma x_2 =$ 195 | $\Sigma x_3 =$ 122 | $\Sigma x_4 =$ 90 | $\Sigma x_5 =$ 161 | T = 695 | $\Sigma x_1^2 =$ 14371 | $\Sigma x_2^2 =$ 13623 | $\Sigma x_3^2 =$ 15586 | $\Sigma x_4^2 =$ 5612 | $\Sigma x_5^2 =$ 12753 |

Step(1) N = 20 Step(2) T = 695 Step(3) $T^2/N = \frac{(695)^2}{20} = 24151.25$

Step(4)

$$TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 + \Sigma x_5^2 - T^2/N$$

$$= 14371 + 13623 + 15586 + 5612 + 12753 - 24151.25$$

$$= 37793.75$$

Step(5) $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} + \frac{(\Sigma x_5)^2}{N_1} - T^2/N$

$N_1 \rightarrow$ no. of elements in each column

$$= \frac{(127)^2}{4} + \frac{(195)^2}{4} + \frac{(122)^2}{4} + \frac{(90)^2}{4} + \frac{(161)^2}{4} - 24151.25$$

$= 1613.50$

Step(6) $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - T^2/N$.

$N_2 \rightarrow$ no. of elements in each row.

$$= \frac{(-31)^2}{5} + \frac{(105)^2}{5} + \frac{(157)^2}{5} + \frac{(464)^2}{5} - 24151.25$$

$= 26234.95$

Step(7) $SSE = TSS - SSC - SSR = 37793.75 - 1613.50 - 26234.95$

$SSE = 9945.3$

Step(8) ANOVA TABLE.

| Source of Variation | SS | d.f | MS | Variance ratio | Table value at $\alpha = 5\% = 0.05$ |
|---|---|---|---|---|---|
| B/w Columns | SSC = 1613.50 | $C-1 = 5-1$ $= 4$ | MSC $= \frac{SSC}{C-1}$ $= \frac{1613.50}{4}$ $= 403.375$ | MSE > MSC $F_c = \frac{MSE}{MSC} = \frac{828.775}{403.375}$ ~~403.375~~ $= 2.055$ | $v_1 = 12, v_2 = 4$ $F_c = 5.91$ |
| B/w Rows | SSR = 26234.95 | $r-1 = 4-1$ $= 3$ | $MSR = \frac{SSR}{r-1}$ $= \frac{26234.95}{3}$ $= 8744.98$ | MSR > MSE $F_R = \frac{MSR}{MSE}$ $= \frac{8744.98}{828.775} = 10.55$ | $v_1 = 3, v_2 = 12$ $F_C = 3.49$ |
| Error | SSE = 9945.3 | $N-C-r+1$ $= 20-5-4+1$ $= 12$ | $MSE = \frac{SSC}{N-C-r+1}$ $= \frac{9945.3}{12}$ $= 828.775$ | | |

Conclusion

(i) If cal $F_C$ < table $F_C$ we accept $H_0$.

$2.055 < 5.91$ is True. So we accept $H_0$.

(ii) If cal $F_R$ < table $F_R$ we accept $H_0$

$10.55 < 3.49$ is not true. So we reject $H_0$.

(3) M/J-2014 Four Varieties A, B, C, D of a fertiliser are tested in a randomised block design with 4 replication. The plot yields in pounds are as follows.

| Column Row | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | A (12) | D (20) | C (16) | B (10) |
| 2 | D (18) | A (14) | B (11) | C (14) |
| 3 | B (12) | C (15) | D (19) | A (13) |
| 4 | C (16) | B (11) | A (15) | D (20) |

Analyse the experimental yield.

Soln:-

| Variety | Block 1 ($X_1$) | 2 ($X_2$) | 3 ($X_3$) | 4 ($X_4$) | Total | $X_1^2$ | $X_2^2$ | $X_3^2$ | $X_4^2$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 12 | 14 | 15 | 13 | $\Sigma Y_1 = 54$ | 144 | 196 | 225 | 169 |
| B | 12 | 11 | 11 | 10 | $\Sigma Y_2 = 44$ | 144 | 121 | 121 | 100 |
| C | 16 | 15 | 16 | 14 | $\Sigma Y_3 = 61$ | 256 | 225 | 256 | 196 |
| D | 18 | 20 | 19 | 20 | $\Sigma Y_4 = 77$ | 324 | 400 | 361 | 400 |
| | $\Sigma X_1 = 58$ | $\Sigma X_2 = 60$ | $\Sigma X_3 = 61$ | $\Sigma X_4 = 57$ | $T = 236$ | $\Sigma X_1^2 = 868$ | $\Sigma X_2^2 = 942$ | $\Sigma X_3^2 = 963$ | $\Sigma X_4^2 = 865$ |

$H_0$: There is no significant difference b/w Blocks & b/w varieties

$H_1$: There is a significant difference b/w Blocks & b/w varieties.

Step(1) $N = 16$

Step(2) $T = 236$

Step(3) $\frac{T^2}{N} = \frac{(236)^2}{16} = 3481$

Step(4)

$$TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - \frac{T^2}{N}$$

$$= 868 + 942 + 963 + 865 - 3481$$

$$= 157.$$

Step(5) $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$N_1 \rightarrow$ no. of elements in each row.

$= \frac{(58)^2}{4} + \frac{(60)^2}{4} + \frac{(61)^2}{4} + \frac{(57)^2}{4} - 3481$

$= 2.$

Step(6) $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} - T^2/N$

$= \frac{(54)^2}{4} + \frac{(44)^2}{4} + \frac{(61)^2}{4} + \frac{(77)^2}{4} - 3481$

$= 729 + 484 + 930.25 + 1482.25 - 3481$

$= 144.5$

Step(7) $SSE = TSS - SSC - SSR = \cancel{324}\ 157 - 2 - 144.5$

$= 10.5$

Step(8) ANOVA TABLE

| Source of Variation | SS | d.f | MS | Variance ratio | Table value $\alpha = 5\% = 0.05$ |
|---|---|---|---|---|---|
| B/w columns | $SSC = 2$ | $C-1 = 3$ | $MSC = \frac{SSC}{C-1} = 2/3 = 0.67$ | $MSE > MSC$ $F_C = \frac{MSE}{MSC} = \frac{1.17}{0.67} = 1.75$ | $v_1 = 9,\ v_2 = 3$ $F_C = 8.81$ |
| B/w Rows | $SSR = 144.5$ | $r-1 = 3$ | $MSR = \frac{SSR}{r-1} = \frac{144.5}{3} = 48.17$ | $F_R = \frac{MSR}{MSE} = \frac{48.17}{1.17} = 41.17$ $MSR > MSE$ | $F_R = 3.86$ $(v_1 = 3, v_2 = 9)$ |
| Error | $SSE = 10.5$ | $N-C-r+1 = 16-4-4+1 = 9$ | $MSE = \frac{SSE}{N-C-r+1} = \frac{10.5}{9} = 1.17$ | | |

Step(9)

Conclusion

(i) If cal $F_C <$ table $F_C$, we accept $H_0$.

$1.75 < 8.81$ we accept $H_0$.

(ii) If cal $F_R <$ table $F_R$, we accept $H_0$.

$41.17 < 3.86$ is not True.

So we reject $H_0$.

(3) M/J-2014 Four Varieties A, B, C, D of a fertiliser are tested in a randomised block design with 4 replication. The plot yields in pounds are as follows.

| Column Row | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | A (12) | D (20) | C (16) | B (10) |
| 2 | D (18) | A (14) | B (11) | C (14) |
| 3 | B (12) | C (15) | D (19) | A (13) |
| 4 | C (16) | B (11) | A (15) | D (20) |

Analyse the experimental yield.

Soln:-

| Variety | Block 1 ($X_1$) | 2 ($X_2$) | 3 ($X_3$) | 4 ($X_4$) | Total | $X_1^2$ | $X_2^2$ | $X_3^2$ | $X_4^2$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 12 | 14 | 15 | 13 | $\Sigma Y_1 = 54$ | 144 | 196 | 225 | 169 |
| B | 12 | 11 | 11 | 10 | $\Sigma Y_2 = 44$ | 144 | 121 | 121 | 100 |
| C | 16 | 15 | 16 | 14 | $\Sigma Y_3 = 61$ | 256 | 225 | 256 | 196 |
| D | 18 | 20 | 19 | 20 | $\Sigma Y_4 = 77$ | 324 | 400 | 361 | 400 |
| | $\Sigma X_1 = 58$ | $\Sigma X_2 = 60$ | $\Sigma X_3 = 61$ | $\Sigma X_4 = 57$ | $T = 236$ | $\Sigma X_1^2 = 868$ | $\Sigma X_2^2 = 942$ | $\Sigma X_3^2 = 963$ | $\Sigma X_4^2 = 865$ |

$H_0$: There is no significant difference b/w Blocks & b/w varieties

$H_1$: There is a significant difference b/w Blocks & b/w varieties.

Step(1) $N = 16$

Step(2) $T = 236$

Step(3) $T^2/N = \frac{(236)^2}{16} = 3481$

Step(4)

$$TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - T^2/N$$

$$= 868 + 942 + 963 + 865 - 3481$$

$$= 157.$$

Step(5) $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$N_1 \rightarrow$ no. of elements in each row.

$= \frac{(58)^2}{4} + \frac{(60)^2}{4} + \frac{(61)^2}{4} + \frac{(57)^2}{4} - 3481$

$= 2.$

Step(6) $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - T^2/N$

$= \frac{(54)^2}{4} + \frac{(44)^2}{4} + \frac{(61)^2}{4} + \frac{(77)^2}{4} - 3481$

$= 729 + 484 + 930.25 + 1482.25 - 3481$

$= 144.5$

Step(7) $SSE = TSS - SSC - SSR = \cancel{324}\ 157 - 2 - 144.5$

$= 10.5$

Step(8) ANOVA TABLE

| Source of Variation | SS | d.f | MS | Variance ratio | Table value $\alpha = 5\% = 0.05$ |
|---|---|---|---|---|---|
| B/w Columns | SSC = 2 | $C-1 = 3$ | $MSC = \frac{SSC}{C-1}$ $= 2/3 = 0.67$ | MSE > MSC $F_C = \frac{MSE}{MSC} = \frac{1.17}{0.67}$ $= 1.75$ | $V_1 = 9,\ V_2 = 3$ $F_C = 8.81$ |
| B/w Rows | SSR = 144.5 | $r-1 = 3$ | $MSR = \frac{SSR}{r-1}$ $= \frac{144.5}{3}$ $= 48.17$ | $F_R = \frac{MSR}{MSE}$ $= \frac{48.17}{1.17} = 41.17$ MSR > MSE | $F_R = 3.86$ $(V_1 = 3, V_2 = 9)$ |
| Error | SSE = 10.5 | $N-C-r+1$ $= 16-4-4+1$ $= 9$ | $MSE = \frac{SSE}{N-C-r+1}$ $= \frac{10.5}{9}$ $= 1.17$ | | |

Step(9)

Conclusion

(i) If cal $F_C$ < table $F_C$, we accept $H_0$.

$1.75 < 8.81$ we accept $H_0$.

(ii) If cal $F_R$ < table $F_R$, we accept $H_0$.

$41.17 < 3.86$ is not True.

So we reject $H_0$.

(4.) The following data represent the number of units production per day turned out by different workers, using 4 different types of machines.

M/J-2013
D/M-2011

| | | Machine type | | | |
|---|---|---|---|---|---|
| | | A | B | C | D |
| Workers | 1 | 44 | 38 | 47 | 36 |
| | 2 | 46 | 40 | 52 | 43 |
| | 3 | 34 | 36 | 44 | 32 |
| | 4 | 43 | 38 | 46 | 33 |
| | 5 | 38 | 42 | 49 | 39 |

Test whether the five men differ with respect to mean productivity and test whether the mean productivity is the same for the 4 different machine types.

Soln:- Subtract 40 from each value

| Workers | Machine Type | | | | Total | $X_1^2$ | $X_2^2$ | $X_3^2$ | $X_4^2$ |
|---|---|---|---|---|---|---|---|---|---|
| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | | | | | |
| $Y_1$ | 4 | −2 | 7 | −4 | $\Sigma Y_1 = 5$ | 16 | 4 | 49 | 16 |
| $Y_2$ | 6 | 0 | 12 | 3 | $\Sigma Y_2 = 21$ | 36 | 0 | 144 | 9 |
| $Y_3$ | −6 | −4 | 4 | −8 | $\Sigma Y_3 = -14$ | 36 | 16 | 16 | 64 |
| $Y_4$ | 3 | −2 | 6 | −7 | $\Sigma Y_4 = 0$ | 9 | 4 | 36 | 49 |
| $Y_5$ | −2 | 2 | 9 | −1 | $\Sigma Y_5 = 8$ | 4 | 4 | 81 | 1 |
| | $\Sigma X_1 = 5$ | $\Sigma X_2 = -6$ | $\Sigma X_3 = 38$ | $\Sigma X_4 = -17$ | $T = 20$ | $\Sigma X_1^2 = 101$ | $\Sigma X_2^2 = 28$ | $\Sigma X_3^2 = 326$ | $\Sigma X_4^2 = 139$ |

$H_0$: There is no significant difference b/w machines & b/w workers

$H_1$: There is a significant difference b/w machines & b/w workers.

Step(1) $N = 20$

Step(2) $T = 20$

Step(3) $T^2/N = \frac{(20)^2}{20} = 20$

Step(4) $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - T^2/N$

$= 101 + 28 + 326 + 139 - 20$

$= 574$

Step(5) $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - T^2/N$

$= \frac{(5)^2}{5} + \frac{(-6)^2}{5} + \frac{(38)^2}{5} + \frac{(-17)^2}{5} - 20$

$= 338.8$

Step(6) $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} - T^2/N + \frac{(\Sigma y_5)^2}{N_2}$

$= \frac{(5)^2}{4} + \frac{(21)^2}{4} + \frac{(-14)^2}{4} + \frac{0^2}{4} + 8^2/4 - 20$

$= 161.5$

Step(7)

$SSE = TSS - SSC - SSR = 574 - 338.8 - 161.5 = 73.7$

Step(8) ANOVA TABLE.

| Source of Variation | SS | d.f | MS | Variance ratio | Table value $\alpha = 0.05$ |
|---|---|---|---|---|---|
| B/w Columns | SSC = 338.8 | $C-1$ $= 3$ | $MSC = \frac{SSC}{C-1}$ $= \frac{338.8}{3}$ $= 112.933$ | $MSC > MSE$ $F_C = \frac{MSC}{MSE}$ $= \frac{112.933}{6.142} = 18.38$ | $v_1 = 3, v_2 = 12$ $F_C = 3.49$ |
| B/w Rows | SSR = 161.5 | $r-1$ $= 4$ | $MSR = \frac{SSR}{r-1}$ $= \frac{161.5}{4}$ $= 40.375$ | $MSR > MSE$ $F_C = \frac{MSR}{MSE}$ $= \frac{40.375}{6.142} = 6.574$ | $v_1 = 4, v_2 = 12$ $F_R = 3.26$. |
| Error | SSE = 73.7 | $N-(C-r+1)$ $= 12$ | $MSE = \frac{SSE}{N-(C-r+1)}$ $= \frac{73.7}{12} = 6.142$ | | |

Conclusion :-

(i) If cal $F_C <$ table $F_C$ we accept $H_0$.

$18.38 < 3.49$ is not True. So we reject $H_0$.

(ii) If cal $F_R <$ table $F_R$ we accept $H_0$.

$6.574 < 3.26$ is not true. So we reject $H_0$.

# Latin Square Design

1 The following is a Latin square of a design, when 4 varieties of seeds are being tested. Set up a sample analysis of variance table and state your conclusion. You may carry out suitable change of origin and scale.

M/J-2013
N/D-2013
A/M-2017

| | | | |
|---|---|---|---|
| A 105 | B 95 | C 125 | D 115 |
| C 115 | D 125 | A 105 | B 105 |
| D 115 | C 95 | B 105 | A 115 |
| B 95 | A 135 | D 95 | C 115 |

Solution

Subtract 100 and then divide by 5

| | | | |
|---|---|---|---|
| A 1 | B −1 | C 5 | D 3 |
| C 3 | D 5 | A 1 | B 1 |
| D 3 | C −1 | B 1 | A 3 |
| B −1 | A 7 | D −1 | C 3 |

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | Total | $X_1^2$ | $X_2^2$ | $X_3^2$ | $X_4^2$ |
|---|---|---|---|---|---|---|---|---|---|
| $Y_1$ | 1 | −1 | 5 | 3 | $\Sigma Y_1 = 8$ | 1 | 1 | 25 | 9 |
| $Y_2$ | 3 | 5 | 1 | 1 | $\Sigma Y_2 = 10$ | 9 | 25 | 1 | 1 |
| $Y_3$ | 3 | −1 | 1 | 3 | $\Sigma Y_3 = 6$ | 9 | 1 | 1 | 9 |
| $Y_4$ | −1 | 7 | −1 | 3 | $\Sigma Y_4 = 8$ | 1 | 49 | 1 | 9 |
| | $\Sigma X_1 =$ 6 | $\Sigma X_2 =$ 10 | $\Sigma X_3 =$ 6 | $\Sigma X_4 =$ 10 | $T =$ 32 | $\Sigma X_1^2 =$ 20 | $\Sigma X_2^2 =$ 76 | $\Sigma X_3^2 =$ 28 | $\Sigma X_4^2 =$ 28 |

$H_0$ : There is no significant difference b/w rows, columns and treatments

$H_1$ : There is a significant difference b/w rows, columns and treatments.

Step(1) $N = 16$

Step(2) $T = 32$

Step(3) $T^2/N = \frac{(32)^2}{16} = 64$.

Step(4) $TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - T^2/N$

$= 20 + 76 + 28 + 28 - 64$

$= 88$

Step(5) $SSC = \frac{(\Sigma X_1)^2}{N_1} + \frac{(\Sigma X_2)^2}{N_1} + \frac{(\Sigma X_3)^2}{N_1} + \frac{(\Sigma X_4)^2}{N_1} - T^2/N$

$= \frac{(6)^2}{4} + \frac{(10)^2}{4} + \frac{(6)^2}{4} + \frac{(10)^2}{4} - 64 = 4$

Step(6) $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} - T^2/N$

$= \frac{(8)^2}{4} + \frac{(10)^2}{4} + \frac{(6)^2}{4} + \frac{(8)^2}{4} - 64$

$= 2.$

Step(7) To find SSK

Arrange the elements in the order of Treatments

| | | | | | Total |
|---|---|---|---|---|---|
| A | 1 | 1 | 3 | 7 | 12 |
| B | −1 | 1 | 1 | −1 | 0 |
| C | 5 | 3 | −1 | 3 | 10 |
| D | 3 | 5 | 3 | −1 | 10 |

$SSK = \frac{(12)^2}{4} + \frac{(0)^2}{4} + \frac{(10)^2}{4} + \frac{(10)^2}{4} - T^2/N$

$= 36 + 0 + 25 + 25 - 64$

$= 22.$

Step(8) $SSE = TSS - SSC - SSR - SSK = 88 - 4 - 2 - 22 = 60$

Step(9) ANOVA TABLE.

| Source of Variation | SS | d.f | MS | Variance ratio | Table value at 5% |
|---|---|---|---|---|---|
| B/w columns | $SSC = 4$ | $k-1 = 4-1 = 3$ | $MSC = \frac{SSC}{k-1} = \frac{4}{3} = 1.33$ | $MSE > MSC$ <br> $\text{Cal } F_C = \frac{MSE}{MSC} = 7.52$ | $v_1 = 6, v_2 = 3$ <br> $F_C = 8.94$ |
| B/w Rows | $SSR = 2$ | $k-1 = 3$ | $MSR = \frac{SSR}{k-1} = \frac{2}{3} = 0.67$ | $MSE > MSR$ <br> $\text{Cal } F_R = \frac{MSE}{MSR} = 14.9$ | $v_1 = 6, v_2 = 3$ <br> $F_R = 8.94$ |
| B/w Treatments (K) | $SSK = 22$ | $k-1 = 3.$ | $MSK = \frac{SSK}{k-1} = 7.33$ | $MSE > MSK$ <br> $F_T = \frac{MSE}{MSK} = 1.36$ | $F_T = 8.94.$ |
| Error | $SSE = 60$ | $(k-1)(k-2) = (4-1)(4-2) = 6.$ | $MSE = \frac{SSK}{(k-1)(k-2)} = 10$ | | |

Step(10) Cal $F_C <$ Table $F_C$. ⇒ we ~~reject~~ accept Ho for columns

Cal $F_R >$ Table $F_R$ ⇒ Reject Ho for rows

Cal $F_T <$ Table $F_T$. ⇒ Accept Ho for Treatments.

(2) A Variable trial was conducted on wheat with 4 varieties in a Latin Square Design. The plan of the experiment and the per plot yield are given below 

M/J-2012
N/D-2016
A/M-2017

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| C | 25 | B | 23 | A | 20 | D | 20 |
| A | 19 | D | 19 | C | 21 | B | 18 |
| B | 19 | A | 14 | D | 17 | C | 20 |
| D | 17 | C | 20 | B | 21 | A | 15 |

Analyse the data and interpret the result.

Soln :- Subtract 20 from all entries.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| C | 5 | B | 3 | A | 0 | D | 0 |
| A | −1 | D | −1 | C | 1 | B | −2 |
| B | −1 | A | −6 | D | −3 | C | 0 |
| D | −3 | C | 0 | B | 1 | A | −5 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Total | $x_1^2$ | $x_2^2$ | $x_3^2$ | $x_4^2$ |
|---|---|---|---|---|---|---|---|---|---|
| $y_1$ | 5 | 3 | 0 | 0 | $\Sigma y_1 = 8$ | 25 | 9 | 0 | 0 |
| $y_2$ | −1 | −1 | 1 | −2 | $\Sigma y_2 = -3$ | 1 | 1 | 1 | 4 |
| $y_3$ | −1 | −6 | −3 | 0 | $\Sigma y_3 = -10$ | 1 | 36 | 9 | 0 |
| $y_4$ | −3 | 0 | 1 | −5 | $\Sigma y_4 = -7$ | 9 | 0 | 1 | 25 |
| | $\Sigma x_1 = 0$ | $\Sigma x_2 = -4$ | $\Sigma x_3 = -1$ | $\Sigma x_4 = -7$ | $T = -12$ | $\Sigma x_1^2 = 36$ | $\Sigma x_2^2 = 46$ | $\Sigma x_3^2 = 11$ | $\Sigma x_4^2 = 29$ |

$H_0$ : There is no significant difference b/w rows and colums and Treatment

$H_1$ : There is a significant difference b/w rows and columns and Treatment

Step(1) $N = 16$

Step(2) $T = -12$

Step(3) $T^2/N = \frac{144}{16} = 9$

Step(4)

$$TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - T^2/N$$

$$= 36 + 46 + 11 + 29 - 9$$

$$= 113$$

Step(5) $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - T^2/N$

$= 0^2/4 + (-4)^2/4 + (-1)^2/4 + (-7)^2/4 - 9$

$= 7.5$

Step(6) $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} - T^2/N$

$= 8^2/4 + (-3)^2/4 + (-10)^2/4 + (-7)^2/4 - 9$

$= 46.5$

Step(7) SSK

| | | | | | Total |
|---|---|---|---|---|---|
| A | 0 | −1 | −6 | −5 | −12 |
| B | 3 | −2 | −1 | 1 | 1 |
| C | 5 | 1 | 0 | 0 | 6 |
| D | 0 | −1 | −3 | −3 | −7 |

$SSk = \frac{(-12)^2}{4} + (1)^2/4 + (6)^2/4 + (-7)^2/4 - T^2/N$

$= \frac{144}{4} + 1/4 + 36/4 + 49/4 - 9$

$= 48.5$

Step(8) $SSE = TSS - SSC - SSR - SSK = 113 - 7.5 - 46.5 - 48.5 = 10.5$

Step(9) ANOVA Table

| Source of Variation | SS | d.f | MS | Variance ratio | Table Value $\alpha = 5\% = 0.05$ |
|---|---|---|---|---|---|
| B/w columns | SSC = ~~40~~ 7.5 | K−1 = 3 | MSC $= \frac{7.5}{3} = 2.5$ | MSC > MSE $F_C = \frac{MSE}{MSE} = \frac{2.5}{1.75} = 1.43$ | $v_1 = 3, v_2 = 6$ $F_C = 4.76$ |
| B/w Rows | SSR = ~~40~~ 46.5 | K−1 = 3 | $MSR = \frac{46.5}{3} = 15.5$ | MSR > MSE $F_R = \frac{MSR}{MSE} = 8.86$ | $v_1 = 3, v_2 = 6$ $F_R = 4.76$ |
| B/w Treatments (K) | SSK = 48.5 | K−1 = 3 | MSK $= \frac{48.5}{3} = 16.17$ | MSK > MSE $F_T = \frac{MSK}{MSE} = 9.24$ | $v_1 = 3, v_2 = 6$ $F_T = 4.76$ |
| Error | SSE = 10.5 | (K−1)(K−2) = 3×2 = 6 | $MSE = \frac{10.5}{6} = 1.75$ | | |

Step(9)

Cal $F_C$ < table $F_C$
Cal $F_R$ > table $F_R$
Cal $F_T$ > table $F_T$

$\Rightarrow$ We accept Ho for raw and reject Ho for column and Treatments.

(3) N/D-2012 A farmer wishes to test the effects of four different fertilizers. A, B, C and D on the yield of wheat. In order to eliminate sources of error due to variability in soil fertility, he uses the fertilizers, in a Latin square arrangement as indicated in the following table, where the numbers indicate yields in bushels per unit area

| | | | |
|---|---|---|---|
| A 18 | C 21 | D 25 | B 11 |
| D 22 | B 12 | A 15 | C 19 |
| B 15 | A 20 | C 23 | D 24 |
| C 22 | D 21 | B 10 | A 17 |

Perform an analysis of variance to determine, if there is a significant difference b/w the fertilizers at $\alpha = 0.05$ level of significance.

Soln

Subtract 20 from each value.

| | | | |
|---|---|---|---|
| A −2 | C 1 | D 5 | B −9 |
| D 2 | B −8 | A −5 | C −1 |
| B −5 | A 0 | C 3 | D 4 |
| C 2 | D 1 | B −10 | A −3 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Total | $x_1^2$ | $x_2^2$ | $x_3^2$ | $x_4^2$ |
|---|---|---|---|---|---|---|---|---|---|
| $Y_1$ | −2 | 1 | 5 | −9 | $\Sigma Y_1 = -5$ | 4 | 1 | 25 | 81 |
| $Y_2$ | 2 | −8 | −5 | −1 | $\Sigma Y_2 = -12$ | 4 | 64 | ~~64~~ 25 | 1 |
| $Y_3$ | −5 | 0 | 3 | 4 | $\Sigma Y_3 = 2$ | 25 | 0 | 9 | 16 |
| $Y_4$ | 2 | 1 | −10 | −3 | $\Sigma Y_4 = -10$ | 4 | 1 | 100 | 9 |
| | $\Sigma x_1 = -3$ | $\Sigma x_2 = -6$ | $\Sigma x_3 = -7$ | $\Sigma x_4 = -9$ | $T = -25$ | $\Sigma x_1^2 = 37$ | $\Sigma x_2^2 = 66$ | $\Sigma x_3^2 = 159$ | $\Sigma x_4^2 = 107$. |

Step(1) $N = 16$

Step(2) $T = -25$

Step(3) $T^2/N = \frac{625}{16} = 39.06$

Step(4)

$$TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - T^2/N$$

$$= 37 + 66 + 159 + 107 - 39.06$$

$$= 329.94$$

Step(5)

$$SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_2} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - T^2/N$$

$$= 9/4 + 36/4 + 49/4 + 81/4 - 39.06 = 4.69$$

Step(6) $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - T^2/N$

$$= \frac{25}{4} + \frac{144}{4} + \frac{4}{4} + \frac{100}{4} - 39.06$$

$$= 29.19$$

Step(7) To find SSK

| | | | | | Total |
|---|---|---|---|---|---|
| A | -2 | -5 | 0 | -3 | -10 |
| B | -9 | -8 | -5 | -10 | -32 |
| C | 1 | -1 | 3 | 2 | 5 |
| D | 5 | 2 | 4 | 1 | 12 |

$$SSK = \frac{(-10)^2}{4} + \frac{(-32)^2}{4} + \frac{(5)^2}{4} + \frac{(12)^2}{4.} - T^2/N$$

$$= \frac{100}{4} + \frac{1024}{4} + \frac{25}{4} + \frac{144}{4} - 39.06$$

$$= 284.19$$

Step(8) $SSE = TSS - SSC - SSR - SSK = 329.94 - 4.69 - 29.19 - 284.19$

$SSE = 11.87$

Step(9) ANOVA Table

| Source of Variation | SS | d.f | MS | Variance ratio | Table value $\alpha = 0.05$ |
|---|---|---|---|---|---|
| B/w columns | SSC $= 4.69$ | $K-1$ $=3$ | $MSC = \frac{SSC}{K-1}$ $= 1.56$ | MSE > MSC $F_c = \frac{MSE}{MSC} = 1.26$ | $v_1 = 6, v_2 = 3$ $F_c = 8.94$ |
| B/w Rows | SSR $= 29.19$ | $K-1 = 3$ | $MSR = \frac{SSR}{K-1}$ $= 9.73$ | $F_R = \frac{MSR}{MSE} = 4.91$ ($\because$ MSR > MSE) | $v_1 = 3, v_2 = 6$ $F_R = 4.76$ |
| B/w Treatments (K) | SSK $= 284.19$ | $K-1 = 3$ | $MSK = \frac{SSK}{K-1}$ $= 94.73$ | $F_T = \frac{MSK}{MSE}$ $= 47.8$ (MSK > MSE) | $v_1 = 3, v_2 = 6$ $F_T = 4.76$ |
| Error | SSE $= 11.87$ | $(K-1)(K-2)$ $= 6$ | $MSE = \frac{SSE}{(K-1)(K-2)}$ $= 1.98$ | | |

Conclusion :-

Cal $F_C$ < table $F_C$ $\Rightarrow$ we accept $H_0$ for column

Cal $F_R$ > table $F_R$ We reject $H_0$ for row and Treatments.

Cal $F_T$ > table $F_T$

# UNIT-III

## Solution of Equations and Eigen Value Problems.

### Solution of Equations:

The following are important types of Equations

Type (i) Algebraic Equation:-

The polynomial $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$

is equal to zero means algebraic eqn.

(a) $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_0 = 0.$

Examples

(1) $5x^7 + 3x^2 + 7x + 16 = 0$

(2) $2x^3 - 3x - 6 = 0.$

Type (ii) Transcendental Equation:-

Equations which are not purely algebraic are called transcendental equation.

(a) If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc, then $f(x) = 0$ is called a transcendental eqn.

Examples:-

(1) $2x^2 + e^x - 5 = 0$

(2) $x + \cos x + 2 = 0$

(3) $\log_{10} x - 5 = 0.$

## Notes

* Every algebraic eqn has atleast one root and $n^{th}$ degree eqn has exactly n roots which are real, imaginary and complex.

* A transcendental eqn may have no root or any number of roots. The roots of this eqn may be real or imaginary.

## Methods of Finding accurate Roots :-

1) Method of Fixed Point (Fixed Point Iteration)

2) Newton - Raphson Method.

## Fixed Point Iteration :-

1) Find the interval of ~~f(x)~~ $(a,b)$ whose roots of $f(x)=0$ lie.

2) Convert $f(x)=0$ into $x = g(x)$

3) Check whether $|g'(x)| < 1 \ \forall x \in (a,b)$

Suppose not this method is not applicable.

4) Choose $x_0$ be any number lie b/w a & b.

Find $1^{st}$ approximation of root, $x_1 = g(x_0)$

$2^{nd}$ approximation of root, $x_2 = g(x_1)$

⋮

$n^{th}$ approximation of root, $x_n = g(x_{n-1})$

Then the sequence $\{x_0, x_1, x_2, \ldots x_n\}$ converge to the root of $f(x)=0$.

# Problems

① Find the real root of the eqn $x^3+x^2-100=0$.

<u>Soln</u>

Let $f(x) = x^3+x^2-100$.

<u>Aim</u> :- To find root of $f(x)=0$.

(i) To find the Location of roots :

$$f(0) = -100 = -ve$$

$$f(1) = 1+1-100 = -ve$$

$$f(2) = 8+4-100 = -ve$$

$$f(3) = 36-100 = -ve$$

$$f(4) = 64+16-100 = -ve$$

$$f(5) = 125+25-100 = +ve.$$

$f(x)$ changes value $-ve$ to $+ve$ on $(4,5)$.

$\therefore$ $f(x)$ has a root which lies b/w $(4,5)$.

The given eqn is $x^3+x^2-100=0$.

$$\Rightarrow x^3+x^2=100$$

$$\Rightarrow x(x^2+x)=100$$

$$\Rightarrow x^2(x+1)=100$$

$$\Rightarrow x^2 = \frac{100}{(x+1)}$$

$$\Rightarrow x = \frac{100}{\sqrt{x+1}} \longrightarrow ①$$

Compare $x = g(x)$ with ①

$$g(x) = \frac{10}{\sqrt{x+1}}.$$

$$\therefore\ g'(x) = 10\, x^{-1/2}\,(x+1)^{-3/2}\cdot(1)$$

$$= \frac{-5}{(x+1)^{3/2}}.$$

$$\therefore\ |g'(x)| = 5/(x+1)^{3/2}.$$

Now $|g'(4)| = 5/5^{3/2} = \frac{1}{\sqrt{5}} < 1$

Also $|g'(5)| = \frac{5}{6^{3/2}} < 1.$

$\therefore$ This method is applicable.

Let $x_0 = 4.2$

1st approximation of root $\Big\} = x_1 = g(x_0) = \frac{10}{\sqrt{x_0+1}} = \frac{10}{\sqrt{4.2+1}} = 4.38529$

$$x_2 = g(x_1) = \frac{10}{\sqrt{x_1+1}} = \frac{10}{\sqrt{4.38529+1}} = 4.30919.$$

$$x_3 = g(x_2) = \frac{10}{\sqrt{4.30919+1}} = 4.33996.$$

Similarly

$x_4 = 4.32744$

$x_5 = 4.33252$

$x_6 = 4.33046$

$x_7 = 4.33129$

$x_8 = 4.33096$

$x_9 = 4.33109$

$x_{10} = 4.33104$

$x_{11} = 4.33106$

$x_{12} = 4.33105$

$x_{13} = 4.33105$

Here $x_{12} = x_{13} = 4.33015$ correct to 5 decimal places.

Hence, the better approximate root is $4.33105$.

② Find a real root of the eqn $\cos x = 3x - 1$ correct to 5 decimal places by fixed point iteration method.

Soln:-

Given $\cos x = 3x - 1$.

$\Rightarrow \cos x - 3x + 1 = 0$.

Take $f(x) = \cos x - 3x + 1 = 0$.

Aim:- To find roots of $f(x) = 0$.

$$\Rightarrow x = \frac{1}{3}(1 + \cos x) = g(x).$$

(i) $g(x) = \frac{1}{3}(1 + \cos x)$.

$$g'(x) = -\tfrac{1}{3}\sin x.$$

$$|g'(x)| = \frac{1}{3}\sin x.$$

To find the Location of root:

$f(0) = 1 - 0 + 1 = 2 = +ve$

$f(1) = \cos 1 - 3 + 1 = -1.4597 = -ve$.

So $f(x)$ has a root lies b/w $(0,1)$.

$|g'(0)| = 0 < 1$

$|g'(1)| = \tfrac{1}{3}\sin 1 < 1$.

So this method can be applied.

Let $x_0 = 0.6$

$x_1 = \frac{1}{3}[1 + \cos(0.6)] = 0.60845$.

$x_2 = \frac{1}{3}[1 + \cos x_1] = \tfrac{1}{3}[1 + \cos(0.60845)] = 0.60684$

$x_3 = \tfrac{1}{3}[1 + \cos(0.60684)] = 0.60715$

$$x_4 = \frac{1}{3}\left[1+\cos x_3\right] = \frac{1}{3}\left[1+\cos(0.60715)\right] = 0.60709$$

$$x_5 = \frac{1}{3}\left[1+\cos x_4\right] = \frac{1}{3}\left[1+\cos(0.60709)\right] = 0.60710.$$

$$x_6 = \frac{1}{3}\left[1+\cos x_5\right] = \frac{1}{3}\left[1+\cos(0.60710)\right] = 0.60710.$$

Hence $x_5 = x_6 = 0.60710$.

Hence, the better approximate root is $0.60710$.

③ Solve $e^x - 3x = 0$ by method of fixed point iteration. (correct to 4 decimal places).

Soln

Given $f(x) = e^x - 3x = 0.$

$f(0) = 1 = +ve$

$f(1) = e - 3 = -ve.$

So $f(x)$ has a root lie b/w the interval $(0,1)$.

Given $e^x - 3x = 0.$

$$\Rightarrow x = \frac{e^x}{3} \Rightarrow \boxed{g(x) = e^x/3}$$

$$|g'(x)| = e^x/3.$$

$$|g'(0)| = 1/3 < 1$$

$$|g'(1)| = e/3 < 1.$$

$\therefore$ This method can be applied.

Let $x_0 = 0.6$

$$x_1 = \frac{1}{3}e^{0.6} = 0.6074.$$

$$x_2 = \frac{1}{3} e^{x_1} = \frac{1}{3} e^{0.6074} = 0.6119$$

$$x_3 = \frac{1}{3} e^{x_2} = \frac{1}{3} e^{0.6119} = 0.6146$$

$$x_4 = \frac{1}{3} e^{x_3} = \frac{1}{3} e^{0.6146} = 0.6163$$

$$x_5 = \frac{1}{3} e^{x_4} = \frac{1}{3} e^{0.6163} = 0.6174$$

$$x_6 = \frac{1}{3} e^{x_5} = \frac{1}{3} e^{0.6174} = 0.6180$$

$$x_7 = \frac{1}{3} e^{0.6180} = 0.6184$$

$$x_8 = \frac{1}{3} e^{0.6184} = 0.6187.$$

$$x_9 = \frac{1}{3} e^{0.6187} = 0.6188$$

$$x_{10} = \frac{1}{3} e^{0.6188} = 0.6189$$

$$x_{11} = \frac{1}{3} e^{0.6189} = 0.6190$$

$$x_{12} = \frac{1}{3} e^{0.6190} = 0.6190.$$

Hence $x_{11} = x_{12} = 0.6190$ correct to 4 decimal places.

Hence the better approximate root is 0.6190.

## Solution of Linear System of Equations

A system of m linear eqns in 'n' unknowns $x_1, x_2, \ldots, x_n$ is a set of eqns of the form

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

This canbe written as

$$AX = B$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} \ldots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

Solution of this system is X

There are 2 methods to find X.

1) Direct method
2) Indirect method.

| 1) Direct method :- | 2) Indirect Method |
|---|---|
| (i) Gauss Elemination Method<br>(ii) Gauss - Jordan method. | (i) Gauss Jacobi Iteration.<br>(ii) Gauss Seidal - Iteration. |

## Augment Matrix:-

If $AX = B$ is a given system of linear eqns then the matrix $[A, B]$ is called augment matrix.

For example,

$$\left.\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ -3x_1 + 2x_2 - 3x_3 &= -6 \\ 2x_1 - 5x_2 + 4x_3 &= 5 \end{aligned}\right\} \text{ is given}$$

Then the above system of eqns can be written as

$$\begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 5 \end{pmatrix}$$

Here $A \quad X \quad = B$.

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -3 \\ 2 & -5 & 4 \end{pmatrix} \quad \underset{\text{(unknowns)}}{X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \quad ; B = \begin{pmatrix} 1 \\ -6 \\ 5 \end{pmatrix}$$

$$\text{Augment matrix} = \underset{(A \quad , B)}{\left(\begin{array}{ccc:c} 1 & -1 & 1 & 1 \\ 3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{array}\right)}$$

# Gauss Elimination Method:-

1) In this method, we first write the augment matrix $[A, B]$

2) From the first column with non zero component, Select the component with the largest absolute value. This component is called pivot.

3) Rearrange the rows to move the pivot element to the top of First column.

4) Make pivot elt as 1, by deving the first row by pivot and make all other elts in this column to zero.

5) Continuing in this way, we can make A as upper triangular matrix.

Then solution $x$ can be obtained by backward substitution

## Problems

① Solve
$$x+2y+z=3$$
$$2x+3y+3z=10$$
$$3x-y+2z=13$$
by Gauss Elimination method.

Soln

The given system can be written as

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}$$

Comparing to $AX = B$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}.$$

Augment matrix $= [A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$

Pivot

pivot elt is '3'.

$$[A, B] = \left[\begin{array}{ccc|c} 3 & -1 & 2 & 13 \\ 2 & 3 & 3 & 10 \\ 1 & 2 & 1 & 3 \end{array}\right] \quad R_3 \rightarrow \text{interchanging } R_1 \,\&\, R_3$$

$$= \left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 11 & 5 & 4 \\ 0 & 7 & 1 & -4 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow 3R_2 - 2R_1 \\ R_3 \rightarrow 3R_3 - R_1 \end{array}$$

6 9 9 30
6 -2 4 26

3 6 3 9
3 -1 2 13

Pivot

pivot elt is 11

$$= \left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 11 & 5 & 4 \\ 0 & 0 & -24 & -72 \end{array}\right] \quad R_3 \rightarrow 11R_3 - 7R_2$$

$11R_3 = 0 \;\; 77 \;\; 11 \;\; -44$
$7R_2 \;\; 0 \;\; 77 \;\; 35 \;\; 28$

$0 \;\; 0 \;\; -24 \;\; -72$

$$-24z = -72.$$

$$\boxed{z = 72/24 = 3}$$

$$11y + 5z = 4$$

$$11y = 4 - 5z = 4 - 5(3)$$

$$11y = -11$$

$$\boxed{y = -1}$$

$$3x - y + 2z = 3$$

$$3x = 3 + y - 2z$$

$$= 3 - 1 - 2(3)$$

$$3x = 3 - 1 - 6$$

$$3x = -4$$

$$\boxed{x = -4/3}$$

$\Rightarrow$ Soln $\boxed{\begin{array}{l} x = -4/3 \\ y = -1 \\ z = 3 \end{array}}$

Checking: $2(-4/3) + 3(-1) + 3(3)$

$= -8/3 - 3 + 9$

Scanned with CamScanner

(1) Solve $x+2y+z=3$
$2x+3y+3z=10$ by Gauss elimination method
$3x-y+2z=13$.

Soln

The given system of eqns can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

$$A\ X = B.$$

The augment matrix is

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

$R_2 - 2R_1$

$$\begin{array}{rcccc} & 2 & 3 & 3 & 10 \\ & 2 & 4 & 2 & 6 \\ \hline (-) & 0 & -1 & 1 & 4 \end{array}$$

$R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - 3R_1$

$$\begin{array}{rcccc} & 3 & -1 & 2 & 13 \\ & 3 & 6 & 3 & 9 \\ \hline (-) & 0 & -7 & -1 & 4 \end{array}$$

$R_3 \to R_3 - 3R_1$

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$

$$\begin{array}{cccc} 0 & -7 & -1 & 4 \\ 0 & -7 & 7 & 28 \\ \hline 0 & 0 & -8 & -24. \end{array}$$

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$\Rightarrow -8z = -24 \quad \Rightarrow \boxed{z=3}$

$-y+z=4 \quad \Rightarrow z-4=y$

$\Rightarrow \boxed{y=-1}$

$x+2y+z=3 \quad \Rightarrow x = 3-2(-1)-3$

$\boxed{x=2}$

$\therefore$ Soln is $\boxed{\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3 \end{aligned}}$

(2) Solve $2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$ by Gauss elimination method.

Soln

Given system can be written as

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

$$A \quad X = B$$

Augment matrix $[A, B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$.

$R_2 \to 2R_2 - 3R_1$ ; $R_3 \to 2R_3 - R_1$

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{array}\right]$$

$$\begin{array}{cccc} 2 & 8 & 18 & 32 \\ 2 & 1 & 1 & 10 \\ \hline 0 & 7 & 17 & 22 \end{array} \qquad \begin{array}{cccc} 6 & 4 & 6 & 36 \\ 6 & 3 & 3 & 30 \\ \hline 0 & 1 & 3 & 6 \end{array}$$

$R_3 \to R_3 - 7R_2$

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$\Rightarrow \quad -4z = -20 \quad \Rightarrow z = 5$

$\Rightarrow \quad y + 3z = 6 \quad \Rightarrow y = 6 - 15 = -9.$

$\Rightarrow \quad 2x + y + z = 10$

$\Rightarrow \quad 2x = 10 - y - z = 10 + 9 - 5 = 14.$

Soln: $\boxed{\begin{array}{l} x = 7 \\ y = -9 \\ z = 5 \end{array}}$

(3) Solve $2x+y+4z=12$ ; $8x-3y+2z=20$ ; $4x+11y-z=33$

by Gauss elimination method.

Soln Given system can be written as

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

$$A \quad X \quad = B$$

$R_2 - 4R_1$

$$\begin{matrix} 8 & -3 & 2 & 20 \\ 8 & 4 & 16 & 48 \\ \hline 0 & -7 & -14 & -28 \end{matrix}$$

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$ ; $R_3 \to R_3 - 2R_2$

$$\begin{matrix} 4 & 11 & -1 & 33 \\ 4 & 2 & 8 & 24 \\ \hline 0 & 9 & -9 & 9 \end{matrix}$$

$R_3 - 2R_2$

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array}\right]$$

$R_3 \to 7R_3 + 9R_2$

$$\begin{matrix} 0 & 63 & -63 & 63 \\ 0 & -63 & -126 & -252 \\ \hline 0 & 0 & -189 & -189 \end{matrix}$$

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -189 & -189 \end{array}\right]$$

$\Rightarrow \quad -189z = -189 \quad \Rightarrow \boxed{z=1}$

$-7y - 14z = -28 \quad \Rightarrow 7y + 14z = 28$

$\Rightarrow 7y = 28 - 14$

$\Rightarrow \boxed{y = 14/7 = 2}$

$2x + y + 4z = 12$

$\Rightarrow 2x = 12 - y - 4z$

$2x = 12 - 2 - 4 = 6.$

Soln $\boxed{\begin{matrix} x = 3 \\ y = 2 \\ z = 1 \end{matrix}}$

(4) Solve the system of eqns

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

by Gauss elimination method.

Soln

The given system can be written as

$$\begin{pmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ -33 \\ 41 \end{pmatrix}$$

$$AX = B.$$

Augment matrix

$$[A,B] = \left[\begin{array}{ccc|c} \overset{\text{pivot}}{10} & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 0 & \overset{\text{pivot}}{52} & -28 & -188 \\ 0 & -34 & 91 & 341 \end{array}\right] \begin{array}{l} \\ R_2 \to 5R_2 - R_1 \\ R_3 \to 10R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to 52R_3 + 34R_2 \end{array}$$

By Backward Substitution,

$$3780z = 11340 \Rightarrow z = 11340/3780 = 3$$

$$52y - 28z = -188 \Rightarrow \cancel{52y = 28(3) - 188}$$

$$\Rightarrow 52y = -188 + 28(3) = 104 \Rightarrow \boxed{y = 2}$$

$$10x - 2y + 3z = 23$$
$$10x = 23 + 4 - 9 = 10$$
$$\boxed{x = 1}$$

Soln $\boxed{\begin{array}{l} x = 1 \\ y = -2 \\ z = 3 \end{array}}$

Gauss Jordan Method:-

1) In this method, given system of eqns can be written as ~~5x+~~ $Ax = B$.

2) We convert the ~~augment~~ matrix A into diagonal matrix.

3) Then we can calculate the unknowns $X$.

Problems

(1) Solve
$$5x + 2y + z = 12$$
$$x + 4y + 2z = 15 \quad \text{by Gauss Jordan method.}$$
$$x + 2y + 5z = 20.$$

Soln:-

The given system is
$$\begin{pmatrix} 5 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 15 \\ 20 \end{pmatrix}$$

Augment matrix $= \left(\begin{array}{ccc|c} 5 & 2 & 1 & 12 \\ 1 & 4 & 2 & 15 \\ 1 & 2 & 5 & 20 \end{array}\right)$

$$\begin{array}{l} \text{Pivot} \\ = \left(\begin{array}{ccc|c} 5 & 2 & 1 & 12 \\ 1 & 4 & 2 & 15 \\ 1 & 2 & 5 & 20 \end{array}\right) \end{array}$$

$$= \left(\begin{array}{ccc|c} 5 & 2 & 1 & 12 \\ 0 & 18 & 9 & 63 \\ 0 & 8 & 24 & 88 \end{array}\right) \begin{array}{l} R_2 \to 5R_2 - R_1 \\ R_3 \to 5R_3 - R_1 \end{array}$$

$\begin{matrix} 5 & 20 & 10 & 75 \\ 5 & 2 & 1 & 12 \end{matrix}$

$\begin{matrix} 5 & 10 & 25 & 100 \\ 5 & 2 & 1 & 12 \end{matrix}$

$$= \left(\begin{array}{ccc|c} 45 & 0 & 0 & 45 \\ 0 & 18 & 9 & 63 \\ 0 & 0 & 180 & 540 \end{array}\right) \begin{array}{l} R_1 \to 9R_1 - R_2 \\ R_3 \to 9R_3 - 4R_2 \end{array}$$

$\begin{matrix} 45 & 18 & 9 & 108 \\ 0 & 18 & 9 & 63 \end{matrix}$

$\begin{matrix} 0 & 72 & 216 & 792 \\ 0 & 72 & 36 & 252 \end{matrix}$

$$= \left(\begin{array}{ccc|c} 45 & 0 & 0 & 45 \\ 0 & 360 & 0 & 720 \\ 0 & 0 & 180 & 540 \end{array}\right).$$

Here A becomes diagonal matrix.

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\Rightarrow \boxed{\begin{array}{l} x=1 \\ y=2 \\ z=3 \end{array}}$$ is a soln.

(2) Using Gauss – Jordan Method solve.

$$4x_1 + x_2 + x_3 = 6$$
$$x_1 + 4x_2 + x_3 = 6$$
$$x_1 + x_2 + 4x_3 = 6.$$

Soln

$$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

$$A \quad X \quad = B.$$

Augment matrix $[A, B] = \left[\begin{array}{ccc|c} 4 & 1 & 1 & 6 \\ 1 & 4 & 1 & 6 \\ 1 & 1 & 4 & 6 \end{array}\right]$

pivot

$$= \left[\begin{array}{ccc|c} 4 & 1 & 1 & 6 \\ 1 & 4 & 1 & 6 \\ 1 & 1 & 4 & 6 \end{array}\right]$$

pivot

$$= \left[\begin{array}{ccc|c} 4 & 1 & 1 & 6 \\ 0 & 15 & 3 & 18 \\ 0 & 3 & 15 & 18 \end{array}\right] \quad \begin{array}{l} R_2 \to 4R_2 - R_1 \\ R_3 \to 4R_3 - R_1 \end{array}$$

$$\begin{array}{cccc} 4 & 16 & 4 & 24 \\ 4 & 1 & 1 & 6 \end{array}$$

$$\begin{array}{cccc} 4 & 4 & 16 & 24 \\ 4 & 1 & 1 & 6 \end{array}$$

$$= \left[\begin{array}{ccc|c} 4 & 1 & 1 & 6 \\ 0 & 15 & 3 & 18 \\ 0 & 0 & 72 & 72 \end{array}\right] \quad R_3 \to 5R_3 - R_2$$

$$\begin{array}{cccc} 0 & 15 & 75 & 90 \\ 0 & 15 & 3 & 18 \end{array}$$

$$\cancel{= \left[\begin{array}{ccc|c} 60 & 0 & 12 & 72 \\ 0 & 360 & 0 & \end{array}\right] \quad \begin{array}{l} R_1 \to 15R_1 - R_2 \\ R_2 \to 24R_2 - R_3 \end{array}}$$

$$= \left(\begin{array}{ccc|c} 60 & 0 & 12 & 72 \\ 0 & 15 & 3 & 18 \\ 0 & 0 & 72 & 72 \end{array}\right) \quad R_1 \to 15R_1 - R_2$$

$$\cancel{= \left(\begin{array}{ccc|c} 60 & 0 & 0 & 0 \end{array}\right) \quad R_1 \to R_1 - 4R_2}$$

$$= \left(\begin{array}{ccc|c} 360 & 0 & \cancel{12}\ 0 & \cancel{72}\ 360 \\ 0 & 15 & 3 & 18 \\ 0 & 0 & 72 & 72 \end{array}\right) \quad R_1 \to 6R_1 - R_3$$

$$= \left(\begin{array}{ccc|c} 360 & 0 & 0 & 360 \\ 0 & 360 & 0 & 360 \\ 0 & 0 & 72 & 72 \end{array}\right) \quad R_2 \to 24R_2 - R_3$$

Here A becomes diagonal matrix

$$\Rightarrow \quad 360x = 360 \quad \Rightarrow \boxed{x = 1}$$

$$\Rightarrow \quad 360y = 360 \quad \Rightarrow \boxed{y = 1}$$

$$\Rightarrow \quad 72z = 72 \quad \Rightarrow \boxed{z = 1}$$

$\therefore$ Soln is $\boxed{\begin{array}{l} x = 1 \\ y = 1 \\ z = 1 \end{array}}$

# Indirect Method

## Jacobi's Iteration Method :-

Let us consider the system of 3 eqns.

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3.$$

$$\Rightarrow \quad x = \frac{1}{a_1}\left[d_1 - b_1 y - c_1 z\right]$$
$$y = \frac{1}{b_2}\left[d_2 - a_2 x - c_2 z\right]$$
$$z = \frac{1}{c_3}\left[d_3 - a_3 x - b_3 y\right]$$

First fix $x_0 = 0,\ y_0 = 0,\ z_0 = 0$ in above equalities

we get first approximation $x_1 = \frac{1}{a_1}\left(d_1 - b_1 y_0 - c_1 z_0\right) = d_1/a_1$

$$y_1 = d_2/b_2$$
$$z_1 = d_3/c_3$$

Second approximation is sub $x_1, y_1, z_1$ values in above system of equalities.

$$x_2 = \frac{1}{a_1}\left[d_1 - b_1 y_1 - c_1 z_1\right]$$
$$y_2 = \frac{1}{b_2}\left[d_2 - a_2 x_1 - c_2 z_1\right]$$
$$z_2 = \frac{1}{c_3}\left[d_3 - a_3 x_1 - b_3 y_1\right]$$

Continue this process untill $x_n = x_{n+1}$

$$y_n = y_{n+1}$$
$$z_n = z_{n+1}.$$

Then soln is $\boxed{\begin{array}{c} x = x_n \\ y = y_n \\ z = z_n \end{array}}$

## Diagonally Dominant Matrix:-

We say a matrix is diagonally dominant if the numerical value (or absolute value) of leading diagonal element in each row is greater than or equal to the sum of the numerical values of the other elts in that row.

Example

1) $\begin{bmatrix} 5 & 1 & -1 \\ 1 & 4 & 2 \\ 1 & -2 & 5 \end{bmatrix}$ is diagonally dominant

Since $5 > 1+1$

$4 > 1+2$

$5 > 1+2$

2) $\begin{bmatrix} 5 & 1 & -1 \\ 5 & 2 & 3 \\ 1 & -2 & 5 \end{bmatrix}$ is not diagonally dominant.

Since 2 is not greater than $(5+3)$

i.e.) $2 < (5+3)$

# Gauss Seidal Iteration Method:-

* First check the matrix A is diagonally dominant or not.

* If A is not diagonally dominant then by chant it to diagonally dominant by interchanging its rows.

* Find $x^{(1)}$ by put $y=0$ & $z=0$.

Find $y^{(1)}$ by put $z=0$ & using $x^{(1)}$ values.

Find $z^{(1)}$ value using $x^{(1)}$ & $y^{(1)}$ values.

* Find $x^{(2)}$: using $y^{(1)}$ & $z^{(1)}$

Find $y^{(2)}$ using $x^{(2)}$ & $z^{(1)}$

Find $z^{(2)}$ using $x^{(2)}$ & $y^{(2)}$

Continue in this process, until $x^{(n)} = x^{(n+1)}$
$y^{(n)} = y^{(n+1)}$
$z^{(n)} = z^{(n+1)}$

Then soln is $x = x^{(n)}$
$y = y^{(n)}$
$z = z^{(n)}$

Problem

(1) Solve the following system of eqns

$$27x + 6y - z = 85$$

$$x + y + 54z = 110$$

$$6x + 15y + 2z = 72.$$

using (i) Gauss - Jacobi method

(ii) Gauss - Seidal method.

Soln

(i) Given system of eqns can be written as

$$\begin{pmatrix} 27 & 6 & -1 \\ 1 & 1 & 54 \\ 6 & 15 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 85 \\ 110 \\ 72 \end{pmatrix}$$

$$A \quad X = B.$$

Here A is not diagonally dominant.

$\therefore$ We interchange second & third eqns.

$$\Rightarrow \left.\begin{aligned} 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \\ x + y + 54z &= 110 \end{aligned}\right\} \text{ is correct form.}$$

$$\Rightarrow x = \frac{1}{27}\left[85 - 6y + z\right]$$

$$y = \frac{1}{15}\left[72 - 6x - 2z\right]$$

$$z = \frac{1}{54}\left[110 - x - y\right]$$

First Iteration :-

Put $y = z = 0$

$$x_1 = 85/27 = 3.148$$

Put $x = z = 0$.

$$y_1 = 72/15 = 4.8$$

Put $x = y = 0$

$$z_1 = \frac{110}{54} = 2.037.$$

Second Iteration :-

$$x_2 = \frac{1}{27}\left[85 - 6(3.148) + 2.037\right] = 2.157.$$

$$y_2 = \frac{1}{15}\left[72 - 6(3.148) - 2(2.037)\right] = 3.269$$

$$z_2 = \frac{1}{54}\left[110 - 3.148 - 3.269\right] = 1.890.$$

Continuing in this manner

$$x_9 = x_{10} = 2.426$$

$$y_9 = y_{10} = 3.573$$

$$z_9 = z_{10} = 1.926.$$

$\therefore$ $\boxed{\begin{aligned} x &= 2.426 \\ y &= 3.573 \\ z &= 1.926 \end{aligned}}$ correct to 3 decimal places.

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(ii) Gauss - Seidal Iteration :-

First Iteration :-

Put $y = z = 0$

$$x^{(1)} = 85/27 = 3.148.$$

Put $z = 0$, $y^{(1)} = \frac{1}{15}\left[72 - 6(3.148) - 2(0)\right] = 3.541$

$$z^{(1)} = \frac{1}{54}\left(110 - 3.148 - 3.541\right) = 1.913$$

Second Iteration :-

$$x^{(2)} = \frac{1}{27}\left[85 - 6(3.541) + 1.913\right] = 2.432$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6(2.432) - 2(1.913)\right] = 3.572$$

$$z^{(2)} = \frac{1}{54}\left[110 - 2.432 - 3.572\right] = 1.926$$

Third Iteration :-

$$x^{(3)} = \frac{1}{27}\left[85 - 6(3.572) + 1.926\right] = 2.426$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(3)} = \frac{1}{54}\left[110 - 2.426 - \cancel{1.926}\ 3.573\right] = 1.926.$$

Fourth Iteration :-

$$x^{(4)} = \frac{1}{27}\left[85 - 6(3.573) + 1.926\right] = 2.426$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573$$

$$z^{(4)} = \frac{1}{54}\left[110 - 2.426 - 3.573\right] = 1.926.$$

Here $x^{(3)} = x^{(4)}$, $y^{(3)} = y^{(4)}$, $z^{(3)} = z^{(4)}$. Hence Soln is $\boxed{x = 2.426,\ y = 3.573,\ z = 1.926}$

(2) Using Gauss-Seidal iteration method solve the following system of eqns

$$x + 3y + 10z = 24$$
$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35.$$

correct to 3 decimal places.

Soln.

Given $28x + 4y - z = 32 \Rightarrow x = \frac{1}{28}\left[32 - 4y + z\right] \rightarrow (1)$

$2x + 17y + 4z = 35 \Rightarrow y = \frac{1}{17}\left[35 - 2x - 4z\right] \rightarrow (2)$

$x + 3y + 10z = 24 \Rightarrow z = \frac{1}{10}\left[24 - x - 3y\right]. \rightarrow (3)$

First Iteration :-

Put $y = 0 = z$ in (1)

$$x = \frac{1}{28}(32) = 1.143$$

Put $x = 1.143$, $z = 0$ in (2)

$$y_1 = \frac{1}{17}\left[35 - 2(1.143)\right] = 1.924.$$

Put $x = 1.143$, $y = 1.924$ in (3)

$$z = \frac{1}{10}\left[24 - 1.143 - 3(1.924)\right] = 1.709.$$

Second Iteration :-

Put $z = 1.709$, $y = 1.924$ in (1)

$$x_2 = \frac{1}{28}\left[32 - 4(1.924) + 1.709\right] = 0.929$$

$$y_2 = \frac{1}{17}\left[35 - 2(0.929) - 4(1.709)\right] = 1.547.$$

$$z_2 = \frac{1}{10}\left[24 - 0.929 - 3(1.547)\right] = 1.843$$

Third Iteration

$$x_3 = \frac{1}{28}\left[32 - 4(1.547) + 1.843\right] = 0.989.$$

$$y_3 = \frac{1}{17}\left[35 - 2(0.989) - 4(1.843)\right] = 1.509$$

$$z_3 = \frac{1}{10}\left[24 - 0.989 - 3(1.509)\right] = 1.848.$$

Fourth Iteration:-

$$x_4 = \frac{1}{28}\left[32 - 4(1.509) + 1.848\right] = 0.850$$

$$y_4 = \frac{1}{17}\left[35 - 2(0.850) - 4(1.848)\right] = 1.524$$

$$z_4 = \frac{1}{10}\left[24 - 0.850 - 3(1.524)\right] = 1.858.$$

Fifth Iteration:-

$$x_5 = \frac{1}{28}\left[32 - 4(1.524) + 1.858\right] = 0.992$$

$$y_5 = \frac{1}{17}\left[35 - 2(0.992) - 4(1.858)\right] = 1.504$$

$$z_5 = \frac{1}{10}\left[24 - 0.992 - 3(1.504)\right] = 1.845$$

Sixth Iteration:-

$$x_6 = \frac{1}{28}\left[32 - 4(1.504) + 1.845\right] = 0.993$$

$$y_6 = \frac{1}{17}\left[35 - 2(0.993) - 4(1.845)\right] = 1.504$$

$$z_6 = \frac{1}{10}\left[24 - 0.993 - 3(1.504)\right] = 1.845$$

$\therefore$ Solution is $x = 0.993$

$y = 1.504$ correct to 3 decimal places.

$z = 1.845$

## Eigen Value of a Matrix by Power Method

Let A be any square matrix.

If $\lambda$ is the eigenvalue of A then $Ax = \lambda x$ where x is some non-zero vector called eigen vector corresponding to $\lambda$.

$$\Rightarrow \quad Ax - \lambda x = 0$$

$$\Rightarrow \quad (A - \lambda I)x = 0.$$

Also $\det(A - \lambda I) = 0$ is called char. eqn.

Roots of $|A - \lambda I| = 0$ are called eigen values of A

* If A is $n \times n$ matrix where n is large it is difficult to find the eigen values of A.

Using Power method, we can calculate the eigen value in such cases.

### Power Method:-

This method, can be applied to find numerically the greatest eigen value of a square matrix (also called dominant eigenvalue).

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of a $n \times n$ matrix A.

Among these, let $\lambda_1$ be the dominant eigen value.

ie, $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$.

If the corresponding eigenvectors are $x_0, x_1, \ldots, x_n$ then any arbitrary vector '$y$' can be written as

$$y = a_0 x_0 + a_1 x_1 + \cdots + a_n x_n.$$

Since the eigenvectors are linearly independent

$$A^k y = A^k ( a_0 x_0 + a_1 x_1 + \cdots + a_n x_n).$$

$$= a_0 \lambda_1^k x_0 + a_1 \lambda_2^k x_1 + \cdots + a_n \lambda_n^k x_n.$$

$$(\because A^k x = \lambda^k x).$$

$$= \lambda_1^k \left[ a_0 x_0 + a_1 \left(\frac{\lambda_2}{\lambda_1}\right)^k + \cdots \right]$$

But $\left| \frac{\lambda_i}{\lambda_1} \right| < 1 \quad (i = 1, 2, \ldots n).$

Hence $A^k y = \lambda_1^k a_0 x_0.$

and $A^{k+1} y = \lambda_1^{k+1} a_0 x_0.$

Hence if $k$ is large, $\lambda_1 = \frac{A^{k+1} y}{A^k y}$ where the division is carried out in the corresponding components.

Notes

* If the eigenvalues of A are $-3, 1, 2$ then $-3$ is dominant
* If the eigenvalues of A are $-4, 1, 4$ then A has no dominant eigenvalue since $|-4| = |4|$.
* The power method will work only if A has a dominant eigenvalue.

Problems :-

① Using power method, find all the eigen values of

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Soln :-

Let $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an appropriate eigen vector.

$$AX_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = 5X_2.$$

$$AX_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{pmatrix} 5.2 \\ 0 \\ 1 \end{pmatrix} = 5.2 \begin{pmatrix} 1 \\ 0 \\ 0.4 \end{pmatrix} = 5.2X_3$$

$$AX_3 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 5.4 \\ 0 \\ 3 \end{pmatrix} = 5.4 \begin{pmatrix} 1 \\ 0 \\ 0.6 \end{pmatrix} = 5.4X_4.$$

$$AX_4 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 5.6 \\ 0 \\ 4 \end{pmatrix} = 5.6 \begin{pmatrix} 1 \\ 0 \\ 0.7 \end{pmatrix} = 5.6X_5$$

$$AX_5 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 5.7 \\ 0 \\ 4.5 \end{pmatrix} = 5.7 \begin{pmatrix} 1 \\ 0 \\ 0.8 \end{pmatrix} = 5.7X_6$$

$$AX_6 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 5.8 \\ 0 \\ 5 \end{pmatrix} = 5.8 \begin{pmatrix} 1 \\ 0 \\ 0.9 \end{pmatrix} = 5.8X_7$$

$$AX_7 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 5.9 \\ 0 \\ 5.5 \end{pmatrix} = 5.9 \begin{pmatrix} 1 \\ 0 \\ 0.9 \end{pmatrix} = 5.8X_8.$$

Hence $X_7 = X_8 = \begin{pmatrix} 1 \\ 0 \\ 0.9 \end{pmatrix}$.

Hence numerically largest eigen value is 6. and the corresponding eigen vector is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Let $\lambda_1 = 6$, $\lambda_2, \lambda_3$ be two other eigen values.

We know that, $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace of } A$

$$6 + \lambda_2 + \lambda_3 = 5 - 2 + 5 = 8.$$

$$\Rightarrow \boxed{\lambda_2 + \lambda_3 = 2} \longrightarrow (1)$$

Also $\lambda_1 \lambda_2 \lambda_3 = |A|$

$$= 5\left[-10\right] + 1\left[0+2\right]$$

$$= -50 + 2$$

$$= -48.$$

(ie) $6 \lambda_2 \lambda_3 = -48.$

$$\Rightarrow \boxed{\lambda_2 \lambda_3 = -8} \longrightarrow (2)$$

From (1) & (2) $\boxed{\begin{matrix}\lambda_2 = 4 \\ \lambda_3 = -2\end{matrix}}$

(2) Find the dominant eigen value and the corresponding eigen vector of $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find also the least latent root and hence the third eigen value also.

Soln:-

Let $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be an appropriate eigen vector.

$$AX_1 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1\, X_2$$

$$AX_2 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.4 \\ 0 \end{pmatrix} = 7\, X_3$$

$$AX_3 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.4 \\ 1.8 \\ 0 \end{pmatrix} = 3.4\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 3.4\, X_4$$

$$AX_4 = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 4\, X_5$$

Here $X_4 = X_5$.

Hence the numerically largest eigenvalue $= 4$ and the corresponding eigen vector $= \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$

To find the least eigenvalue :-

Let $B = A - 4I$ . ($\because \lambda_1 = 4$).

We will find the dominant eigen value of B.

Let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

$$BY_1 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix}$$

$$= -3\,Y_2$$

$$BY_2 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1.6666 \\ 0 \end{pmatrix} = -5\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = -5Y_3$$

$$BY_3 = \begin{pmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 1.6666 \\ 0 \end{pmatrix} = -5\begin{pmatrix} 1 \\ -0.3333 \\ 0 \end{pmatrix}$$

$\therefore$ Dominant eigen value of B is '$-5$'.

i.e) " " " " $(A-4I)$ is $-5$.

$\therefore$ Adding 4, smallest eigen value of A is $-1$.

Sum of eigenvalues $=$ Trace of A

$$= 1 + 2 + 3 = 6.$$

i.e) $4 + (-1) + \lambda_3 = 6$.

$$\lambda_3 = 6 + 1 - 4$$

$$\boxed{\lambda_3 = 3}$$

All the three eigen values are $4, 3, -1$.

Aliter:-

Let $\lambda_1 = -5$ and $\lambda_2, \lambda_3$ be other 2 eigen values.

We know that,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 3$$

$$4 \cancel{-5} + \lambda_2 + \lambda_3 = 6.$$

$$\boxed{\lambda_2 + \lambda_3 = 2} \longrightarrow ①$$

$$\lambda_1 \lambda_2 \lambda_3 = |A|$$

$$= 1\left[6\right] - 6\left[3\right] + 1\left[0\right]$$

$$= 6 - 18$$

$$4(\lambda_2 \lambda_3) = -12.$$

$$\boxed{\lambda_2 \lambda_3 = -3} \longrightarrow ②$$

From ① & ②.

$$\boxed{\begin{array}{l}\lambda_2 = 3 \\ \lambda_3 = -1\end{array}}$$

$\therefore$ All three eigen values are $4, 3, -1$.

## Eigen Value of a Matrix by Jacobi Method :-

Let $A$ be a given real symmetric matrix. Its eigen values of $A$ are real and $\exists$ a real orthogonal matrix $B$ such that $B^{-1}AB$ is a diagonal matrix.

Jacobi's method consists of diagonalising $A$ by applying a series of orthogonal transformations $B_1, B_2, \ldots, B_r$ such that their product $B$ satisfies the eqn $D = B^{-1}AB$.

### Rotation Matrix :-

If $P(x,y)$ is any point in the $xy$-plane and if $OP$ is rotated in the clockwise direction through an angle $\theta$, then the new position $P(x',y')$ is given by

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta.$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}$$

where $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Hence $P$ is called a Rotation matrix in $xy$-plane.

Here $P$ is also an orthogonal matrix, since $PP^T = I$.

Eigen Value of a $2\times 2$ real Symmetric matrix :-

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a symmetric matrix

Here $a_{21} = a_{12}$.

Step(1) :- Assume $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be most general orthogonal rotation matrix.

Let $B = P^T A P$ be the similar transformation.

$\Rightarrow$ B is also symmetric.

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow b_{11} = a_{11}\cos^2\theta + a_{12}\sin 2\theta + a_{22}\sin^2\theta .$$

$$b_{12} = b_{21} = \frac{1}{2}\left[\left(a_{22} - a_{11}\right)\right]\sin 2\theta + a_{12}\cos 2\theta .$$

$$b_{22} = a_{11}\sin^2\theta - a_{12}\sin 2\theta + a_{22}\cos^2\theta .$$

$\because$ A & B are similar & symmetric matrices

$$a_{11} + a_{22} = b_{11} + b_{22} .$$

Step(2) To make B as a diagonal matrix.

$$\theta = \pi/4 \quad \text{if} \quad a_{11} = a_{22} \quad \text{and} \quad a_{12} > 0 .$$

$$\theta = -\pi/4 \quad \text{if} \quad a_{11} = a_{22} \quad \text{and} \quad a_{12} < 0 .$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}}\right) \quad \text{if} \quad a_{11} \neq a_{22} .$$

Step(3) Write down $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ using Value of $\theta$.

Step(4) Get $D = P^T A P$.

The diagonal elts are eigen values.

## Extension to Higher Order Symmetric Matrices:-

Suppose we want to reduce the off-diagonal numerically largest element $a_{ij}$ in $(a_{ij})_{n\times n}$ matrix into zero.

Take

$$D_1 = \begin{array}{cl} \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cos\theta & \cdots & -\sin\theta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \sin\theta & \cdots & \cos\theta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \rightarrow i^{th} \text{ row} \\ \\ \rightarrow j^{th} \text{ row} \\ \\ \\ \end{array} \\ \begin{array}{cc} \downarrow \; i^{th} \text{ column} & \downarrow \; j^{th} \text{ column} \end{array} & \end{array}$$

$D_1$ is a $n\times n$ matrix whose diagonal elts are 1 and all off diagonal elts are zero except

$$a_{ii} = \cos\theta \;,\quad a_{jj} = \cos\theta \;,\quad a_{ij} = -\cos\theta \;,\quad a_{ji} = \sin\theta .$$

Use $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{ij}}{a_{ii} - a_{jj}}\right)$ if $a_{ii} = a_{jj}$

$$= \tan^{-1}\left(\frac{-2a_{ij}}{(a_{jj} - a_{ii}) + \sqrt{(a_{ii} - a_{jj})^2 + 4a_{ij}^2}}\right) \text{ if } a_{ii} < a_{jj}$$

$$= \tan^{-1}\left(\frac{2a_{ij}}{(a_{ii} - a_{jj}) + \sqrt{(a_{ii} - a_{jj})^2 + 4a_{ij}^2}}\right) \text{ if } a_{ii} > a_{jj}.$$

Now $D_1$ is orthogonal.

$$B_1 = D_1^T A D_1$$

$$B_2 = D_2^T B_1 D_2 .$$

Performing series of such rotation $D_1, D_2, D_3 \ldots$ after $k$ operations, we get

$$B_K = D^T A D.$$

## Problems

(1) Using Jacobi method, find the eigen values and eigen vectors of $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

Soln :-

Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Here $a_{11} = a_{22}$ ; $a_{12} = a_{21} = 1 > 0$.

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Here $\theta = \frac{1}{2} \tan^{-1} \left( \frac{2a_{12}}{a_{11} - a_{22}} \right)$

$$= \frac{1}{2} \tan^{-1} \left( \frac{2}{0} \right)$$

$$= \frac{1}{2} \times \tan^{-1}(\infty)$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{4}}$$

$\therefore$ Rotation matrix $P = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$$D = P^T A P$$

$$= \left\{ \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \right\} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5/\sqrt{2} & 5/\sqrt{2} \\ -3/\sqrt{2} & 3/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}.$$

The eigen values are 5, 3 and the eigen vectors of the matrix are the columns of $P$.

Eigen vectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(2) Find the eigen values and eigen vectors of

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Soln :-

The rotation matrix, $P = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

Select $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11} - a_{33}}\right)$

$= \frac{1}{2}\tan^{-1}\left(\frac{2(-1)}{2-2}\right)$

$= \pi/4.$

$$P = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$D = P^T A P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

$\therefore$ The eigen values are 1, 2, 3.

Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

(3) Find the eigen values and eigen vectors of the matrix

$$A = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$$

Soln

The rotation matrix is $P = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{33}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\sqrt{2}}{1-1}\right)$$

$$\theta = \frac{\pi}{4}.$$

$$\therefore P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$D = P^T A P$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & +1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Again reduce the largest off-diagonal elts $a_{12} = a_{21} = 2$ in D into zero.

Consider the rotation matrix

$$P_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Select $\theta$ so that, $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right)$

$$\boxed{\theta = \pi/4}$$

$$\therefore P_1 = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$D_1 = P_1^T D P.$

$$= \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

After two rotations, A is reduced to diagonal matrix $D_1$.

Hence eigenvalues of A are $5, 1, -1$.

Now $P_2 = PP_1 = \begin{pmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{pmatrix}.$

Hence the corresponding eigen vectors are

$$\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

# 2. INTERPOLATION AND APPROXIMATION

**Lagrange's interpolation formula:**

Let $y=f(x)$ be a fun. which takes the values $y_0, y_1, \dots, y_n$ corresponding to $x = x_0, x_1, \dots, x_n$. Then Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n$$

---

**Problems:**

① Find the polynomial $f(x)$ by using Lagrange's formula & hence find $f(3)$ for

| | | | | |
|---|---|---|---|---|
| $x$ : | 0 | 1 | 2 | 5 |
| $f(x)$ : | 2 | 3 | 12 | 147 |

**Sol:** Here $x_0=0$, $x_1=1$, $x_2=2$, $x_3=5$, $y_0=2$, $y_1=3$, $y_2=12$ & $y_3=147$

By Lagrange's interpolation formula, we have

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2)+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12)+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$=\frac{(x-1)(x-2)(x-5)}{(-10)}(2)+\frac{x(x-2)(x-5)}{4}(3)+\frac{x(x-1)(x-5)}{(-6)}(12)$$

$$+\frac{x(x-1)(x-2)}{60}(147)$$

$$\therefore f(3)=\frac{(3-1)(3-2)(3-5)}{-5}+\frac{3(3-2)(3-5)}{4}(3)+3(3-1)(3-5)(-2)+\frac{3(3-1)(3-2)}{60}(147)$$

$$=\frac{4}{5}-\frac{9}{2}+24+\frac{147}{10}=\frac{8-45+240+147}{10}=\frac{350}{10}=35$$

$$\therefore f(3)=35$$

---

② Find the third degree polynomial $f(x)$ satisfying the following data:

| | | | | |
|---|---|---|---|---|
| $x$ : | 1 | 3 | 5 | 7 |
| $y$ : | 24 | 120 | 336 | 720 |

Sol: Here $x_0=1, x_1=3, x_2=5, x_3=7, y_0=24, y_1=120, y_2=336$ & $y_3=720$.

The Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-3)(x-5)(x-7)}{(-2)(-4)(-6)}(24)+\frac{(x-1)(x-5)(x-7)}{(2)(-2)(-4)}(120)$$

$$+\frac{(x-1)(x-3)(x-7)}{(4)(2)(-2)}(336)+\frac{(x-1)(x-3)(x-5)}{(6)(4)(2)}(720)$$

$$=\frac{-1}{2}\left[x^3-15x^2+71x-105\right]+\frac{15}{2}\left[x^3-13x^2+47x-35\right]-21\left[x^3-11x^2+31x-21\right]$$
$$+15\left[x^3-9x^2+23x-15\right]$$

$$=\left[\frac{-1}{2}+\frac{15}{2}-21+15\right]x^3+\left[\frac{15}{2}-13\left(\frac{15}{2}\right)+21(11)-15(9)\right]x^2$$
$$+\left[\frac{-71}{2}+\frac{15}{2}(47)-21(31)+15(23)\right]x+\frac{105}{2}-\frac{15}{2}(35)+21(21)-15(15)$$

$$=x^3+6x^2+11x+6$$

$$\therefore f(x)=x^3+6x^2+11x+6$$

③ Find the missing term in the following table using Lagrange's interpolation.

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 0 | 1 | 2 | 3 | 4 |
| $y$ : | 1. | 3. | 9. | – | 81. |

Sol: Here $x_0=0, x_1=1, x_2=2, x_3=4, y_0=1, y_1=3, y_2=9, y_3=81$

By Lagrange's interpolation formula

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-1)(x-2)(x-4)}{(-1)(-2)(-4)}(1)+\frac{(x-0)(x-2)(x-4)}{(1)(-1)(-3)}(3)$$

$$+\frac{(x-0)(x-1)(x-4)}{(2)(1)(-2)}(9)+\frac{(x-0)(x-1)(x-2)}{(4)(3)(2)}(81)$$

$$\therefore f(3) = \frac{(2)(1)(-1)}{-8} + \frac{(3)(1)(-1)}{3}(3) + \frac{(3)(2)(-1)}{-4}(9) + \frac{(3)(2)(1)}{(4)(3)(2)}(81)$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4} = \frac{1-12+54+81}{4} = 31$$

$$\therefore f(3) = 31$$

(4) Find the parabola of the form $y = ax^2 + bx + c$ passing through the pts $(0,0)$, $(1,1)$ & $(2,20)$.

Sol: Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $y_0 = 0$, $y_1 = 1$ & $y_2 = 20$.

The Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-2)}{(-1)(-2)}(0) + \frac{(x-0)(x-2)}{(1)(-1)}(1) + \frac{(x-0)(x-1)}{(2)(1)}(20)$$

$$= -x(x-2) + 10x(x-1) = -x^2 + 2x + 10x^2 - 10x$$

$$= 9x^2 - 8x$$

$$\therefore f(x) = y = 9x^2 - 8x$$

(5) The mode of a certain frequency curve $y = f(x)$ is very nearer to $x = 9$ & the values of the frequency density $f(x)$ for $x = 8.9, 9, 9.3$ are respectively $0.30, 0.35$ & $0.25$. Calculate the approximate value of the mode.

Sol: Here $x_0 = 8.9$, $x_1 = 9$, $x_2 = 9.3$, $y_0 = 0.3$, $y_1 = 0.35$ & $y_2 = 0.25$

The Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-9)(x-9.3)}{(-0.1)(-0.4)}(0.3) + \frac{(x-8.9)(x-9.3)}{(0.1)(-0.3)}(0.35) + \frac{(x-8.9)(x-9)}{(0.4)(0.3)}(0.25)$$

$$= 7.5(x^2 - 9.3x - 9x + 83.7) - \frac{0.35}{0.03}(x^2 - 9.3x - 8.9x + 82.77) + \frac{0.25}{0.12}(x^2 - 9x - 8.9x + 80.1)$$

$$= 7.5(x^2 - 18.3x + 83.7) - \frac{35}{3}(x^2 - 18.2x + 82.77) + \frac{25}{12}(x^2 - 17.9x + 80.1)$$

$$= \frac{1}{12}\left[90(x^2 - 18.3x + 83.7) - 140(x^2 - 18.2x + 82.77) + 25(x^2 - 17.9x + 80.1)\right]$$

$$= \frac{1}{12}\left[-25x^2 + 453.5x - 2052.3\right]$$

To get the mode, $f'(x) = 0$ & $f''(x) = -ve$

$\therefore f'(x) = 0 \Rightarrow \frac{1}{12}(-50x + 453.5) = 0 \Rightarrow x = \frac{453.5}{50} = 9.07$ $\quad \therefore x = 9.07$

$f''(x) = \frac{1}{12}(-50)$ $\quad \therefore f''(9.07) = \frac{-50}{12} = -ve$

Hence $f(x)$ is maximum at $x = 9.07$

$\therefore$ Mode is 9.07.

---

⑥ Using Lagrange's formula, prove $y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$ nearly.

Sol: From the answer, we have the table

| | | | | |
|---|---|---|---|---|
| $x$ : | $-5$ | $-3$ | $3$ | $5$ |
| $y$ : | $y_{-5}$ | $y_{-3}$ | $y_3$ | $y_5$ |

Here $x_0 = -5,\ x_1 = -3,\ x_2 = 3,\ x_3 = 5$

$y_0 = y_{-5},\ y_1 = y_{-3},\ y_2 = y_3$ & $y_3 = y_5$

The Lagrange's interpolation formula is

$$y_x = \frac{(x+3)(x-3)(x-5)}{(-2)(-8)(-10)} y_{-5} + \frac{(x+5)(x-3)(x-5)}{(2)(-6)(-8)} y_{-3} + \frac{(x+5)(x+3)(x-5)}{(8)(6)(-2)} y_3 + \frac{(x+5)(x+3)(x-3)}{(10)(8)(2)} y_5$$

$$\therefore y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

$$= \frac{-1}{5} y_{-5} + \frac{1}{2} y_{-3} + y_3 - \frac{3}{10} y_5.$$

$$= -0.2y_{-5} + 0.5y_{-3} + y_3 - 0.3y_5$$

$$= y_3 - 0.2y_{-5} + 0.3y_{-3} + 0.2y_{-3} - 0.3y_5$$

$$= y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

---

Inverse Interpolation:

The process of finding a value of $x$ for the corresponding value of $y$ is called inverse interpolation. In this case, we will take $y$ as independent variable & $x$ as dependent variable & use Lagrange's interpolation formula. Taking $y$ as independent variable.

$$x = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)} x_1 + \cdots + \frac{(y-y_0)(y-y_1)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\cdots(y_n-y_{n-1})} x_n$$

This formula is called inverse interpolation formula.

---

Problems:

① Find the age corresponding to the annuity value 13.6 given the table:

| | | | | | |
|---|---|---|---|---|---|
| Age ($x$) : | 30 | 35 | 40 | 45 | 50 |
| Annuity value ($y$) : | 15.9 | 14.9 | 14.1 | 13.3 | 12.5 |

Sol: Here $x_0 = 30,\ x_1 = 35,\ x_2 = 40,\ x_3 = 45,\ x_4 = 50,\ y_0 = 15.9,\ y_1 = 14.9,\ y_2 = 14.1$

$y_3 = 13.3$ & $y_4 = 12.5$.

The inverse Lagrange's interpolation formula is

$$f(y) = x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)} x_3$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} x_4$$

$$= \frac{(y-14.9)(y-14.1)(y-13.3)(y-12.5)}{(1)(1.8)(2.6)(3.4)}(30) + \frac{(y-15.9)(y-14.1)(y-13.3)(y-12.5)}{(-1)(0.8)(1.6)(2.4)}(35)$$

$$+ \frac{(y-15.9)(y-14.9)(y-13.3)(y-12.5)}{(-1.8)(-0.8)(0.8)(1.6)}(40) + \frac{(y-15.9)(y-14.9)(y-14.1)(y-12.5)}{(-2.6)(-1.6)(-0.8)(0.8)}(45)$$

$$+ \frac{(y-15.9)(y-14.9)(y-14.1)(y-13.3)}{(-3.4)(-2.4)(-1.6)(-0.8)}(50)$$

$$\therefore x(13.6) = \frac{(-1.3)(-0.5)(0.3)(1.1)}{(1.8)(2.6)(3.4)}(30) + \frac{(-2.3)(-0.5)(0.3)(1.1)}{-(0.8)(1.6)(2.4)}(35)$$

$$+ \frac{(-2.3)(-1.3)(0.3)(1.1)}{(1.8)(0.8)(0.8)(1.6)}(40) + \frac{(-2.3)(-1.3)(-0.5)(1.1)}{(-2.6)(1.6)(0.8)(0.8)}(45) + \frac{(-2.3)(-1.3)(-0.5)(0.3)}{(3.4)(2.4)(1.6)(0.8)}(50)$$

$$= \frac{55}{136} - \frac{8855}{2048} + \frac{16445}{768} + \frac{56925}{2048} - \frac{22.425}{10.4448} = 43.14$$

$\therefore x(13.6) = 43$

② Find the value of $\theta$ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ using the table

| | | | |
|---|---|---|---|
| $\theta$ : | $21^\circ$ | $23^\circ$ | $25^\circ$ |
| $f(\theta)$: | 0.3706 | 0.4068 | 0.4433 |

Sol: Take $\theta = x$ & $f(\theta) = y$.
Here $\theta_0 = 21^\circ$, $\theta_1 = 23^\circ$, $\theta_2 = 25^\circ$, $f(\theta_0) = 0.3706$, $f(\theta_1) = 0.4068$ & $f(\theta_2) = 0.4433$

The inverse Lagrange's interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$= \frac{(y-0.4068)(y-0.4433)}{(-0.0362)(-0.0727)}(21) + \frac{(y-0.3706)(y-0.4433)}{(0.0362)(-0.0365)}(23) + \frac{(y-0.3706)(y-0.4068)}{(0.0727)(0.0365)}(25)$$

$$x(0.3887) = \frac{(-0.0181)(-0.0546)}{(-0.0362)(-0.0727)}(21) + \frac{(0.0181)(-0.0546)}{(0.0362)(-0.0365)}(23) + \frac{(0.0181)(-0.0181)}{(0.0727)(0.0365)}(25)$$

$$= 7.8858 + 17.2027 - 3.0865 = 22.002$$

$\therefore \theta(0.3887) = 22.002$

## Divided Differences:

Let the funt. $y=f(x)$ take the values $f(x_0), f(x_1), \dots, f(x_n)$ corresponding to the values $x_0, x_1, \dots, x_n$ of the argument $x$ where $x_1-x_0, x_2-x_1, \dots, x_n-x_{n-1}$ need not necessarily be equal.

The first divided difference of $f(x)$ for the arguments $x_0, x_1$ is

$$\Delta(f(x)) = f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0}. \quad \text{IIIly}, \quad f(x_1,x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \text{ \& so on.}$$

The second divided difference of $f(x)$ for three arguments $x_0, x_1, x_2$ is defined as

$$\Delta^2 f(x) = f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}, \quad f(x_1,x_2,x_3) = \frac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1}$$

& so on.

## Properties of divided differences:

$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(x_0)-f(x_1)}{x_0-x_1} = f(x_1,x_0)$

① The divided differences are symmetrical in all their arguments, that is the value of any difference is independent of the order of the arguments.

② The divided difference (of any order) of the sum or difference of two funs. is equal to the sum or difference of the corresponding separate divided differences. $\Delta(f(x) \pm g(x)) = \Delta f(x) \pm \Delta g(x)$

③ The divided difference of the product of a constant & a funt is equal to the product of the constant & the divided difference of the funt. $\Delta(cf(x)) = \frac{cf(x_1)-cf(x_0)}{x_1-x_0} = c\left(\frac{f(x_1)-f(x_0)}{x_1-x_0}\right) = c\Delta f(x)$

④ The $n^{th}$ divided differences of a poly. of the $n^{th}$ degree are constant.

## Problems:

① Form the divided difference table for the following data:

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 1 | 2 | 4 | 7 | 12 |
| $f(x)$ : | 22 | 30 | 82 | 106 | 206 |

$\Delta^2 f(x) = \frac{\Delta f(x_1) - \Delta f(x_0)}{x_2-x_0}$

Sol: ~~Difference table~~

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 1 | 22 | $\frac{30-22}{2-1}=8$ | | | |
| ~~2~~ | ~~30~~ | $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ | $\frac{26-8}{4-1}=6$ | | |
| ~~4~~ | ~~82~~ | | | $\frac{-3.6-6}{7-1}=-1.6$ | |
| 2 | 30 | $\frac{82-30}{4-2}=26$ | | | $\frac{0.51+1.6}{12-1}=0.1918$ |
| | | | $\frac{8-26}{7-2}=-3.6$ | | |
| 4 | 82 | $\frac{106-82}{7-4}=8$ | | $\frac{1.5+3.6}{12-2}=0.51$ | |
| 7 | 106 | | $\frac{20-8}{12-4}=1.5$ | | |
| | | $\frac{206-106}{12-7}=20$ | | | |
| 12 | 206 | | | | |

② Show that $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = \frac{-1}{abcd}$ $\qquad f(x_0,x_1) = \underset{x_1}{\Delta} f(x_0)$ ; $f(x_0,x_1,x_2) = \underset{x_1,x_2}{\Delta^2} f(x_0)$

Sol: Given $f(a) = \frac{1}{a}$

$$f(a,b) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{a-b}{ab(b-a)} = \frac{-1}{ab}$$

$$f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a} = \frac{\frac{-1}{bc}+\frac{1}{ab}}{c-a} = \frac{-a+c}{abc(c-a)} = \frac{+1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d)-f(a,b,c)}{d-a} = \frac{\frac{+1}{bcd} \mp \frac{1}{abc}}{d-a} = \frac{+a \mp d}{abcd(d-a)} = \frac{-1}{abcd}$$

$$\therefore f(a,b,c,d) = \underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = \frac{-1}{abcd}$$

**Newton's divided difference interpolation formula for unequal intervals:**

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_0,x_1,\ldots,x_n)$$

**Problems:**

① Find $f(x)$ as a poly/. in $x$ for the following data by Newton's divided difference formula

| $x$ : | $-4$ | $-1$ | $0$ | $2$ | $5$ |
|---|---|---|---|---|---|
| $f(x)$ : | $1245$ | $33$ | $5$ | $9$ | $1335$ |

Sol:

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| $-4$ | $1245$ | | | | |
| | | $\frac{33-1245}{-1+4} = -404$ | | | |
| | | | $\frac{-28+404}{0+4} = 94$ | | |
| $-1$ | $33$ | | | $\frac{10-94}{2+4} = -14$ | $\frac{13+14}{5+4} = 3$ |
| | | $\frac{5-33}{0+1} = -28$ | | | |
| | | | $\frac{2+28}{2+1} = 10$ | $\frac{88-10}{5+1} = 13$ | |
| $0$ | $5$ | | | | |
| | | $\frac{9-5}{2-0} = 2$ | | | |
| | | | $\frac{442-2}{5-0} = 88$ | | |
| $2$ | $9$ | | | | |
| | | $\frac{1335-9}{5-2} = 442$ | | | |
| $5$ | $1335$ | | | | |

Newton's divided difference interpolation formula is

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_2,x_3,x_4)$$

Here $x_0 = -4,\ x_1 = -1,\ x_2 = 0,\ x_3 = 2,\ x_4 = 5$

$f(x_0) = 1245$, $f(x_0, x_1) = -404$, $f(x_0, x_1, x_2) = 94$, $f(x_0, x_1, x_2, x_3) = -14$, ...,
$f(x_0, x_1, x_2, x_3, x_4) = 3$.

$$\therefore f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x-0)(-14)$$
$$+ (x+4)(x+1)(x-0)(x-2)(3)$$
$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x^4 + 15x^3 + 12x^2$$
$$- 6x^3 - 30x^2 - 24x$$

$$\therefore f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

(12) Using Newton's divided difference formula find the missing value from the table:

| | | | | | |
|---|---|---|---|---|---|
| $x$ : | 1 | 2 | 4 | 5 | 6 |
| $y$ : | 14 | 15 | 5 | – | 9 |

Sol:

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 1 | 14 | | | |
| | | $\frac{15-14}{2-1} = 1$ | | |
| 2 | 15 | | $\frac{-5-1}{4-1} = -2$ | |
| | | $\frac{5-15}{4-2} = -5$ | | $\frac{\frac{7}{4}+2}{6-1} = \frac{3}{4}$ |
| 4 | 5 | | $\frac{2+5}{6-2} = \frac{7}{4}$ | |
| | | $\frac{9-5}{6-4} = 2$ | | |
| 6 | 9 | | | |

Here $x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$

$f(x_0) = 14$

$f(x_0, x_1) = 1$

$f(x_0, x_1, x_2) = -2$

$f(x_0, x_1, x_2, x_3) = \frac{3}{4}$

Newton's divided difference formula is

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$
$$= 14 + (x-1)(1) + (x-1)(x-2)(-2) + (x-1)(x-2)(x-4)\left(\tfrac{3}{4}\right)$$
$$\therefore f(5) = 14 + 4 + (4)(3)(-2) + (4)(3)(1)\left(\tfrac{3}{4}\right) = 3$$

Newton's forward interpolation formula for equal intervals: (Gregory-Newton forward interpolation formula)

$$P_n(x) = P_n(x_0 + uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+ \dots + \frac{u(u-1)(u-2)\cdots(u-(n-1))}{n!}\Delta^n y_0$$

where $u = \frac{x - x_0}{h}$.

Gregory-Newton backward difference interpolation formula:

$$P_n(x) = P_n(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$
$$+ \dots + \frac{v(v+1)(v+2)\cdots(v+(n-1))}{}\nabla^n y_n \text{ where } v = \frac{x - x_n}{}$$

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Problems:

① Using Newton's forward interpolation formula, find the poly/. $f(x)$ satisfying the following data. Hence evaluate $y$ at $x=5$.

| | | | | |
|---|---|---|---|---|
| $x$: | 4 | 6 | 8 | 10 |
| $y$: | 1 | 3 | 8 | 10 |

Sol: Difference table:

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| 4 | 1 | | | |
| | | $3-1=2$ | | |
| 6 | 3 | | $5-2=3$ | |
| | | $8-3=5$ | | $-3-3=-6$ |
| 8 | 8 | | $2-5=-3$ | |
| | | $10-8=2$ | | |
| 10 | 10 | | | |

Here $x_0=4$ , $h=2$

$y_0=1$ ; $\Delta y_0=2$

$\Delta^2 y_0=3$ ; $\Delta^3 y_0=-6$

There are 4 variables given. Hence the poly/. is of degree 3.

$$y(x)=P_3(x)=y_0+\frac{u}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \quad \text{where } u=\frac{x-x_0}{h}$$

$$\therefore u=\frac{x-4}{2}$$

$$y(x)=P_3(x)=1+\frac{\left(\frac{x-4}{2}\right)}{1!}(2)+\frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{2!}(3)+\frac{\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{3!}(-6)$$

$$=1+x-4+\frac{3}{8}(x-4)(x-6)-\frac{1}{8}(x-4)(x-6)(x-8)$$

$$=-3+x+\frac{3}{8}(x^2-10x+24)-\frac{1}{8}(x^3-10x^2+24x-8x^2+80x-192)$$

$$=\frac{1}{8}\left[-24+8x+3x^2-30x+72-x^3+18x^2-104x+192\right]$$

$$\therefore y(x)=\frac{1}{8}\left[-x^3+21x^2-126x+240\right]$$

$$\therefore y(5)=\frac{1}{8}\left[-5^3+21(5)^2-126(5)+240\right]=1.25$$

② Using Newton's forward interpolation formula find the cubic poly/. which takes places the following values:

| | | | | |
|---|---|---|---|---|
| $x$: | 0 | 1 | 2 | 3 |
| $f(x)$: | 1 | 2 | 1 | 10 |

Evaluate $f(4)$ using Newton's backward formula. Is it the same as obtained from the cubic poly/. found above.

Sol:

Difference table:

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ |
|---|---|---|---|---|
| 0 | 1 | | | |
| | | $2-1=1$ | | |
| 1 | 2 | | $-1-1=-2$ | |
| | | $1-2=-1$ | | $10+2=12$ |
| 2 | 1 | | $9+1=10$ | |
| | | $10-1=9$ | | |
| 3 | 10 | | | |

Here $x_0 = 0,\ h = 1$

$y_0 = 1$ ; $\Delta y_0 = 1$

$\Delta^2 y_0 = -2$ ; $\Delta^3 y_0 = 12$

$x_3 = 3$ ; $y_3 = 10$

$\nabla y_3 = 9$ ; $\nabla^2 y_3 = 10$ ; $\nabla^3 y_3 = 12$

There are 4 variables given. Hence the polyl. is of degree 3.

$$f(x) = y(x) = P_3(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \quad \text{where } u = \frac{x-x_0}{h}$$

$$\therefore u = \frac{x-0}{1} = x$$

$$\therefore f(x) = 1 + \frac{x}{1!}(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$= 1 + x - x^2 + x + 2\left(x(x^2-3x+2)\right) = -x^2 + 2x + 1 + 2x^3 - 6x^2 + 4x$$

$$= 2x^3 - 7x^2 + 6x + 1 \quad \text{— (1)}$$

Newton's backward difference formula is

$$f(x) = y(x) = P_3(x) = y_3 + \frac{v}{1!}\nabla y_3 + \frac{v(v+1)}{2!}\nabla^2 y_3 + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_3 \quad \text{where}$$

$$v = \frac{x-x_n}{h} \qquad \therefore v = \frac{x-x_3}{h} = \frac{x-3}{1} = x-3$$

$$\therefore f(x) = 10 + (x-3)(9) + \frac{(x-3)(x-2)}{2!}(10) + \frac{(x-3)(x-2)(x-1)}{3!}(12)$$

$$= 10 + 9x - 27 + 5(x^2 - 5x + 6) + 2(x^3 - 5x^2 + 6x - x^2 + 5x - 6)$$

$$= -17 + 9x + 5x^2 - 25x + 30 + 2x^3 - 12x^2 + 22x - 12$$

$$= 2x^3 - 7x^2 + 6x + 1 \quad \text{— (2)}$$

From (1) & (2), we have Newton's forward & backward difference polys. are same.

$$\therefore f(4) = 2(4)^3 - 7(4)^2 + 6(4) + 1 = 41$$

(3) From the following data, find $\theta$ at $x = 43$ & $x = 84$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ : | 40 | 50 | 60 | 70 | 80 | 90 |
| $\theta$ : | 184 | 204 | 226 | 250 | 276 | 304 |

Also express $\theta$ in terms of $x$.

Sol:

Since six data are given, $P(x)$ is of degree 5. To find $\theta$ at $x=43$ use forward interpolation & to find $\theta$ at $x=84$, use backward interpolation formula.

Difference table:

| $x$ | $\theta$ | $\Delta\theta$ | $\Delta^2\theta$ | $\Delta^3\theta$ | $\Delta^4\theta$ | $\Delta^5\theta$ |
|---|---|---|---|---|---|---|
| 40 | 184 | | | | | |
| | | 20 | | | | |
| 50 | 204 | | 2 | | | |
| | | 22 | | 0 | | |
| 60 | 226 | | 2 | | 0 | |
| | | 24 | | 0 | | 0 |
| 70 | 250 | | 2 | | 0 | |
| | | 26 | | 0 | | |
| 80 | 276 | | 2 | | | |
| | | 28 | | | | |
| 90 | 304 | | | | | |

Here $x_0=40$, $h=10$

$\theta_0=184$, $\Delta\theta_0=20$

$\Delta^2\theta_0=2$, $\Delta^3\theta_0=0$,

$\Delta^4\theta_0=0$, $\Delta^5\theta_0=0$

$x_5=90$, $\theta_5=304$,

$\nabla\theta_5=28$, $\nabla^2\theta_5=2$,

$\nabla^3\theta_5=0$, $\nabla^4\theta_5=0$,

$\nabla^5\theta_5=0$

Newton's forward difference formula is

$$\theta(x)=P_5(x)=\theta_0+\frac{u}{1!}\Delta\theta_0+\frac{u(u-1)}{2!}\Delta^2\theta_0+\frac{u(u-1)(u-2)}{3!}\Delta^3\theta_0 \quad \text{where } u=\frac{x-x_0}{h}=\frac{x-40}{10}$$

$$=184+\frac{\left(\frac{x-40}{10}\right)}{1!}(20)+\frac{\left(\frac{x-40}{10}\right)\left(\frac{x-50}{10}\right)}{2!}(2)$$

$$=184+2(x-40)+\frac{1}{100}(x-40)(x-50)$$

$$\therefore \theta(43)=184+2(3)+\frac{1}{100}(3)(-7)=184+6-\frac{21}{100}=189.79$$

Newton's backward difference formula is

$$\theta(x)=P_5(x)=\theta_5+\frac{v}{1!}\nabla\theta_5+\frac{v(v+1)}{2!}\nabla^2\theta_5 \quad \text{where } v=\frac{x-x_5}{h}=\frac{x-90}{10}$$

$$=304+\frac{\left(\frac{x-90}{10}\right)}{1!}(28)+\frac{\left(\frac{x-90}{10}\right)\left(\frac{x-80}{10}\right)}{2!}(2)$$

$$=304+\frac{28}{10}(x-90)+\frac{1}{100}(x-90)(x-80)$$

$$\therefore \theta(84)=304+\frac{28}{10}(84-90)+\frac{1}{100}(84-90)(84-80)$$

$$=304+\frac{28}{10}(-6)+\frac{1}{100}(-6)(4)=286.96$$

Now, $\theta(x)=\theta_0+\frac{u}{1!}\Delta\theta_0+\frac{u(u-1)}{2!}\Delta^2\theta_0$

$$=184+\left(\frac{x-40}{10}\right)(20)+\frac{1}{100}(x-40)(x-50)$$

$$=184+2x-80+\frac{x^2}{100}-\frac{90x}{100}+\frac{2000}{100}=104+2x+0.01x^2-0.9x+20$$

$$=0.01x^2+1.1x+124$$

④ From the data given below, find the no/. of students whose weight is between 60 to 70.

| Weight | : 0-40 | 40-60 | 60-80 | 80-100 | 100-120 |
|---|---|---|---|---|---|
| No. of students | : 250 | 120 | 100 | 70 | 50 |

Sol: Difference table:

| x (Weight) | y (No. of students) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| Below 40 | 250 | | | | |
| | | 120 | | | |
| Below 60 | 370 | | -20 | | |
| | | 100 | | -10 | |
| Below 80 | 470 | | -30 | | 20 |
| | | 70 | | 10 | |
| Below 100 | 540 | | -20 | | |
| | | 50 | | | |
| Below 120 | 590 | | | | |

Here $x_0=40$, $h=20$

$y_0=250$

$\Delta y_0=120$

$\Delta^2 y_0=-20$

$\Delta^3 y_0=-10$

$\Delta^4 y_0=20$

Let us calculate the no/. of students whose weight is less than 70.

Newton's forward difference formula is

$$y(x)=y_0+\frac{u}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

where $u=\frac{x-x_0}{h}=\frac{x-40}{20}$

$$\therefore y(x)=250+\left(\frac{x-40}{20}\right)(120)+\frac{\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)}{2!}(-20)+\frac{\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)\left(\frac{x-80}{20}\right)}{3!}(-10)$$
$$+\frac{\left(\frac{x-40}{20}\right)\left(\frac{x-60}{20}\right)\left(\frac{x-80}{20}\right)\left(\frac{x-100}{20}\right)}{4!}(20)$$

$$\therefore y(70)=250+180-7.5+0.625+0.46875=423.59=424$$

No/. of students whose weight is between 60 & 70 $=y(70)-y(60)$

$=424-370=54$

Interpolation with a cubic spline:

① From the following table:

| x : | 1 | 2 | 3 |
|---|---|---|---|
| y : | -8 | -1 | 18 |

Compute $y(1.5)$ & $y'(1)$, using cubic spline.

Sol: Here $h=1$ & $n=2$. Also assume $M_0=0$ & $M_2=0$ we have

$$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}[y_{i-1}-2y_i+y_{i+1}] \text{ for } i=1,2,\dots,(n-1)$$

Put $i=1$, we get

$$M_0+4M_1+M_2=6[y_0-2y_1+y_2]$$

$\Rightarrow 4M_1 = 6[-8-2(-1)+18] = 72$

$\Rightarrow M_1 = 18$. The cubic spline in $x_{i-1} \le x \le x_i$ is given by

$$y(x) = S(x) = \frac{1}{6h}\left[(x_i-x)^3 M_{i-1} + (x-x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i-x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}(x-x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right]$$

Put $i=1$, we get

$$S(x) = \frac{1}{6}\left[(x_1-x)^3 M_0 + (x-x_0)^3 M_1\right] + (x_1-x)\left[y_0 - \frac{1}{6}M_0\right] + (x-x_0)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[(x-1)^3 18\right] + (2-x)[-8] + (x-1)\left[-1-\frac{18}{6}\right]$$

$$= 3(x-1)^3 - 8(2-x) - 4(x-1) = 3[x^3-3x^2+3x-1] - 16 + 8x - 4x + 4$$

$$= 3x^3 - 9x^2 + 9x - 3 - 12 + 4x = 3x^3 - 9x^2 + 13x - 15$$

$\therefore y(1.5) = 3(1.5)^3 - 9(1.5)^2 + 13(1.5) - 15 = -5.625$

$y'(x) = S'(x) = 9x^2 - 18x + 13$

$\therefore y'(1) = 9 - 18 + 13 = 4$

---

(2) Obtain the cubic spline approximation for the func. $y=f(x)$ from the following data, given that $y_0'' = y_3'' = 0$.

| | | | | |
|---|---|---|---|---|
| $x$: | $-1$ | $0$ | $1$ | $2$ |
| $y$: | $-1$ | $1$ | $3$ | $35$ |

Sol: Here $h=1$ & $n=3$. Also given that $M_0 = y_0'' = 0$ & $M_3 = y_3'' = 0$.

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] \text{ for } i = 1, 2, \dots, (n-1)$$

Put $i=1$, we get

$$M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2]$$

$\Rightarrow 4M_1 + M_2 = 6[-1-2(1)+3] = 0 \Rightarrow 4M_1 + M_2 = 0$ —(1)

Put $i=2$, we get

$$M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$$

$\Rightarrow M_1 + 4M_2 = 6[1-2(3)+35] = 180 \Rightarrow M_1 + 4M_2 = 180$ —(2)

(2) $\times 4 \Rightarrow 4M_1 + 16M_2 = 720$ —(3)

(3) $-$ (1) $\Rightarrow 15M_2 = 720 \Rightarrow M_2 = 48$

$\therefore M_1 = -12$

The cubic spline in $x_{i-1} \le x \le x_i$, is given by

$$S(x) = y(x) = \frac{1}{6h}\left[(x_i-x)^3 M_{i-1} + (x-x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i-x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}(x-x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \quad \text{—(1)}$$

Put $i=1$, in (1), the cubic spline, for $-1 \le x \le 0$, is given by

$$y(x) = \frac{1}{6}\left[(x_1-x)^3 M_0 + (x-x_0)^3 M_1\right] + (x_1-x)\left[y_0 - \frac{1}{6}M_0\right] + (x-x_0)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[(0-x)^3(0) + (x+1)^3(-12)\right] + (0-x)[-1] + (x+1)\left[1 - \frac{1}{6}(-12)\right]$$

$$y(x) = -2(x+1)^3 + x + 3(x+1)$$
$$= -2(x^3+3x^2+3x+1) + x + 3x + 3$$
$$= -2x^3 - 6x^2 - 6x - 2 + x + 3x + 3 = -2x^3 - 6x^2 - 2x + 1$$

Put $i=2$ in ①, the cubic spline, for $0 \le x \le 1$, is given by

$$y(x) = \frac{1}{6}\left[(x_2-x)^3 M_1 + (x-x_1)^3 M_2\right] + (x_2-x)\left[y_1 - \frac{1}{6}M_1\right] + (x-x_1)\left[y_2 - \frac{1}{6}M_2\right]$$
$$= \frac{1}{6}\left[(1-x)^3(-12) + (x-0)^3(48)\right] + (1-x)\left[1 + \frac{12}{6}\right] + (x-0)\left[3 - \frac{48}{6}\right]$$
$$= -2(1-x)^3 + 8x^3 + 3(1-x) - 5x$$
$$= -2(1-3x+3x^2-x^3) + 8x^3 + 3 - 3x - 5x$$
$$= -2 + 6x - 6x^2 + 2x^3 + 8x^3 + 3 - 8x = 10x^3 - 6x^2 - 2x + 1$$

Put $i=3$ in ①, the cubic spline, for $1 \le x \le 2$, is given by

$$y(x) = \frac{1}{6}\left[(x_3-x)^3 M_2 + (x-x_2)^3 M_3\right] + (x_3-x)\left(y_2 - \frac{1}{6}M_2\right) + (x-x_2)\left(y_3 - \frac{1}{6}M_3\right)$$
$$= \frac{1}{6}\left[(2-x)^3(48)\right] + (2-x)\left(3 - \frac{48}{6}\right) + (x-1)(35-0)$$
$$= 8(2-x)^3 - 5(2-x) + 35(x-1)$$
$$= 8(8-12x+6x^2-x^3) - 10 + 5x + 35x - 35$$
$$= 64 - 96x + 48x^2 - 8x^3 + 40x - 45$$
$$= -8x^3 + 48x^2 - 56x + 19$$

Hence the required cubic spline approximation for the given funl. is

$$y(x) = \begin{cases} -2x^3 - 6x^2 - 2x + 1, & \text{for } -1 \le x \le 0 \\ 10x^3 - 6x^2 - 2x + 1, & \text{for } 0 \le x \le 1 \\ -8x^3 + 48x^2 - 56x + 19, & \text{for } 1 \le x \le 2 \end{cases}$$

---

Conditions for Cubic Spline:

① The funl. values must be equal at the interior knots.

② The first & last funl. must pass through the end pts.

③ The first derivatives at the interior knots must be equal.

④ The second derivatives at the interior knots must be equal.

⑤ The second derivatives at the end knots are zero.

---

## Numerical Single Integrations :-

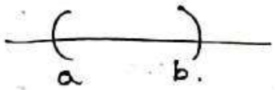
There are 2 methods for evaluate single integrations using Numerical methods

(i) Trapezoidal Rule

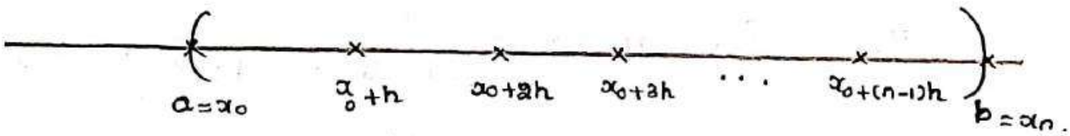
(ii) Simpsons $\frac{1}{3}$ Rule.

### (i) Trapezoidal Rule :-

Suppose we want to find $\int_a^b f(x)\,dx$.

Consider the interval 


Divide the interval into $n$ sub intervals with equal width '$h$'.




Take $a=x_0$, $x_0+h=x_1$, $x_0+2h=x_2$, .... $x_0+(n-1)h=x_{n-1}$, $b=x_n$.

Let $y_0, y_1, y_2, \ldots, y_{n-1}, y_n$ be the values of $f(x)$ at the corresponding points $x_0, x_1, x_2, \ldots, x_n$ respectively.

Then $\int_a^b f(x)\,dx = \frac{h}{2}\Big[$(Sum of first & last ordinates) + 2 (Sum of remaining ordinates)$\Big]$

(OR)

$$\int_a^b f(x)\,dx = \frac{h}{2}\Big[(y_0+y_n) + 2\,(y_1+y_2+\cdots+y_{n-1})\Big]$$

## Problems

(1) Evaluate the integral $\int_1^2 \frac{1}{1+x^2}\,dx$, using Trapezoidal Rule with two sub intervals.

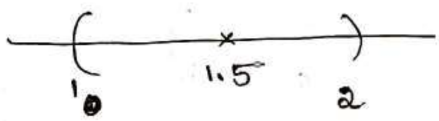
Soln:-

$$f(x) = \frac{1}{1+x^2}$$

$$a=1, \quad b=2.$$

$$h = \frac{2-1}{2} = 0.5.$$

∴ we devide the enterval $(1, 2)$ as follows.




Take $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$.

Now we will find the values of $y_0, y_1, y_2$.

$$y_0 = f(x) \text{ at } x_0 = \frac{1}{1+x_0^2} = \frac{1}{1+1^2} = 0.5$$

$$y_1 = f(x) \text{ at } x_1 = \frac{1}{1+x_1^2} = \frac{1}{1+1.5^2} = 0.3077.$$

$$y_2 = f(x) \text{ at } x_2 = \frac{1}{1+2^2} = \frac{1}{5} = 0.2.$$

∴ By Trapezoidal Rule

$$\int_1^2 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0+y_2) + 2y_1\right]$$

$$= \frac{0.5}{2}\left[(0.5+0.2) + 2(0.3077)\right]$$

$$= 0.3289.$$

② Dividing the range into 10 equal parts, find the value of $\int_0^{\pi/2} \sin x \, dx$ by Trapezoidal rule.

Soln :-

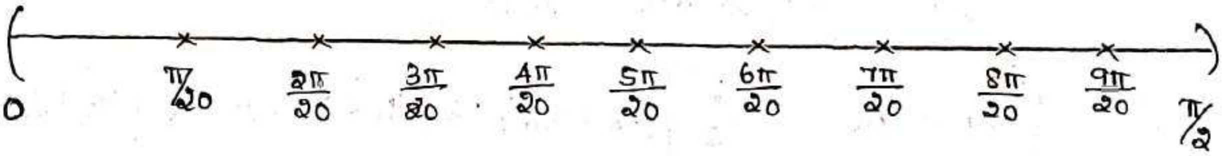
$f(x) = \sin x$

$a = 0 \quad , \quad b = \pi/2 .$

$h = \frac{b-a}{10} = \frac{\pi/2 - 0}{10}$

$h = \pi/20 .$

Now we devide the enterval $(0, \pi/2)$ into 10 sub-interval with equal width $h = \pi/20$.




Take $x_0 = 0$, $x_1 = \pi/20$, $x_2 = 2\pi/20$, $x_3 = 3\pi/20$, $x_4 = 4\pi/20$

$x_5 = 5\pi/20$, $x_6 = 6\pi/20$, $x_7 = \frac{7\pi}{20}$, $x_8 = \frac{8\pi}{20}$, $x_9 = \frac{9\pi}{20}$, $x_{10} = \pi/2$.

Now we calculate the values of $y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}$.

$y_0 = \sin 0 = 0$

$y_1 = \sin(\pi/20) = 0.1564$

$y_2 = \sin\left(\frac{2\pi}{20}\right) = 0.3090$

$y_3 = \sin\left(\frac{3\pi}{20}\right) = 0.4540$

$$y_4 = \sin\left(\frac{4\pi}{20}\right) = 0.5878$$

$$y_5 = \sin\left(\frac{5\pi}{20}\right) = 0.7071$$

$$y_6 = \sin\left(\frac{6\pi}{20}\right) = 0.8090$$

$$y_7 = \sin\left(\frac{7\pi}{20}\right) = 0.8910$$

$$y_8 = \sin\left(\frac{8\pi}{20}\right) = 0.9511$$

$$y_9 = \sin\left(\frac{9\pi}{20}\right) = 0.9877$$

$$y_{10} = \sin\left(\frac{10\pi}{20}\right) = 1.$$

$\therefore$ By Trapezoidal Rule,

$$\int_0^{\pi/2} \sin x \, dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{\pi}{20 \times 2}\left[(0+1) + 2(0.1564 + 0.3090 + 0.4540 + 0.5878 + 0.7071 + 0.8090 + 0.8910 + 0.9511 + 0.9877)\right]$$

$$= \frac{\pi}{40}(12.7062)$$

$$= 0.9980.$$

## Simpson's 1/3 Rule :-

Suppose we want to find $\int_a^b f(x)\,dx$.

consider the interval $(a, b)$.

Here we must divide $(a,b)$ into even number of subintervals n with equal width $h = \frac{b-a}{n}$.

Take the values

$x_0 = a$, $x_0 + h = x_1$, $x_0 + 2h = x_2, \ldots,$ $x_0 + nh = b$.

Calculate the corresponding 'y' values

$y_0, y_1, y_2, y_3, \ldots, y_n$.

Then
$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\Big].$$

### Problems :-

1. Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ by simpson's rule. Also compare this by actual soln.

Soln :-

$$f(x) = \frac{1}{1+x^2};$$

$$a = 0, \quad b = 6.$$

Here we divide $(a,b)$ into 6 subintervals.

$$h = \frac{b-a}{6} = \frac{6-0}{6} = 1.$$


Take $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

$x_4 = 4$, $x_5 = 5$, $x_6 = 6$.

Now

$$y_0 = \frac{1}{1+0^2} = 1$$

$$y_1 = \frac{1}{1+1^2} = 0.5$$

$$y_2 = \frac{1}{1+2^2} = 0.2$$

$$y_3 = \frac{1}{1+3^2} = 0.1$$

$$y_4 = \frac{1}{1+4^2} = 0.058824$$

$$y_5 = \frac{1}{1+5^2} = 0.038462$$

$$y_6 = \frac{1}{1+6^2} = 0.027027.$$

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{3}\left[(1+0.027027) + 2(0.2+0.058824) + 4(0.5+0.1+0.038462)\right]$$

$$= 1.36617433$$

Actual Integration:-

$$\int_0^6 \frac{1}{1+x^2}\,dx = \Big[\tan^{-1}(x)\Big]_{x=0}^{x=6} = \tan^{-1}(6) - \tan^{-1}(0)$$

$$= 1.40564765$$

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(2) Evaluate $\int_0^5 \frac{1}{4x+5}\,dx$ by simpson's $\frac{1}{3}$ rule

with $n=10$. Hence find the value of $\log_e 5$.

Soln

$$f(x) = \frac{1}{4x+5}.$$

$$a=0, \quad b=5.$$

$$h = \frac{b-a}{10} = \frac{5-0}{10} = 0.5.$$

$\therefore$ We divide the interval $(0,5)$ into 10 sub-interval

with equal width 0.5.

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$. $x_4 = 2$, $x_5 = 2.5$

$x_6 = 3$, $x_7 = 3.5$, $x_8 = 4$, $x_9 = 4.5$, $x_{10} = 5$.

Now we find the corresponding $y$ values.

$$y_0 = \frac{1}{4(0)+5} = 0.2$$

$$y_1 = \frac{1}{4(0.5)+5} = 0.1429$$

$$y_2 = \frac{1}{4(1)+5} = 0.1111$$

$$y_3 = \frac{1}{4(1.5)+5} = 0.0909$$

$$y_4 = \frac{1}{4(2)+5} = 0.0769$$

$$y_5 = \frac{1}{4(2.5)+5} = 0.0667$$

$$y_6 = \frac{1}{4(3)+1} = 0.0588$$

$$y_7 = \frac{1}{4(3.5)+1} = 0.0526$$

$$y_8 = \frac{1}{4(4)+1} = 0.0476$$

$$y_9 = \frac{1}{4(4.5)+1} = 0.0434$$

$$y_{10} = \frac{1}{4(5)+1} = 0.04 .$$

By simpsons $\frac{1}{3}$ Rule,

$$\int_0^5 \frac{1}{4x+5}\,dx = \frac{h}{3}\Big[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\Big]$$

$$= \frac{0.5}{3}\Big[(0.2+0.04) + 2(0.1111+0.0769+0.0588+0.0476) + 4(0.1429+0.0909+0.0667+0.0526+0.0434)\Big]$$

$$= 0.4025$$

Actual integration :-

$$\int_0^5 \frac{1}{4x+5}\,dx = \frac{1}{4}\Big[\log(4x+5)\Big]_{x=0}^{x=5} = \frac{1}{4}\Big[\log 25 - \log 5\Big]$$

$$= \frac{1}{4}\log 5$$

$$\therefore \frac{1}{4}\log 5 = 0.4025$$

$$\Rightarrow \log 5 = 4 \times 0.4025$$

$$= 1.61$$

 

# NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

**Defn.:**

An equation which involves the differential coefficients of one variable is called the ordinary differential eqn..

**Taylor's Series method:**

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$$

$$\Rightarrow y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots$$

**Problems:**

① Using Taylor's series method find $y$ at $x=0.1$ if $\frac{dy}{dx} = x^2y-1$, $y(0)=1$.

Sol: Given $y'=x^2y-1$, $x_0=0$, $y_0=1$. ~~W.k.t, x₁=0.1~~

Taylor's series formula is

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$y'=x^2y-1$ $\qquad$ $y_0' = x_0^2y_0 - 1 = -1$

$y''=x^2y'+y.2x$ $\qquad$ $y_0'' = x_0^2y_0' + 2x_0y_0 = 0$

$y''' = x^2y''+2xy'+2xy'+2y = x^2y''+4xy'+2y$ $\qquad$ $y_0''' = x_0^2y_0''+4x_0y_0'+2y_0 = 2$

$y^{iv} = x^2y'''+2xy''+4xy''+4y'+2y'$ $\qquad$ $y_0^{iv} = x_0^2y_0'''+6x_0y_0''+6y_0' = -6$

$= x^2y'''+6xy''+6y'$

$$\therefore y(0.1) = 1 + (0.1-0)(-1) + \frac{(0.1-0)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(-6) + \cdots$$

$$= 1 - 0.1 + 0.00033 - 0.000025 = 0.9003$$

② Solve $y'=x+y$, $y(0)=1$. by Taylor's series method. Find the values of $y$ at $x=0.1$ & $0.2$.

Sol: Given $y'=x+y$, $x_0=0$, $y_0=1$,

Taylor's series formula is

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$$\Rightarrow y(x) = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \cdots$$

$y' = x+y \qquad y_0' = x_0 + y_0 = 1$

$y'' = 1+y' \qquad y_0'' = 1 + y_0' = 2$

$y''' = y'' \qquad y_0''' = y_0'' = 2$

$y^{iv} = y''' \qquad y_0^{iv} = y_0''' = 2$

$$\therefore y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) + \dots = 1.1103$$

$$y(0.2) = 1 + (0.2)(1) + \frac{(0.2)^2}{2!}(2) + \frac{(0.2)^3}{3!}(2) + \frac{(0.2)^4}{4!}(2) + \dots = 1.2428$$

---

③ Solve $\frac{dy}{dx} = y^2 + x^2$ with $y(0) = 1$. Find the values of $y$ at $x = 0.1, 0.2 \,\&\, 0.4$ by using Taylor's series method.

**Sol:** Given $y' = y^2 + x^2$, $x_0 = 0$, $y_0 = 1$

Taylor's series formula is

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \dots$$

$$= y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \dots$$

$y' = y^2 + x^2 \qquad y_0' = y_0^2 + x_0^2 = 1$

$y'' = 2yy' + 2x \qquad y_0'' = 2y_0 y_0' + 2x_0 = 2$

$y''' = 2yy'' + 2y'^2 + 2 \qquad y_0''' = 2y_0 y_0'' + 2y_0'^2 + 2 = 8$

$y^{iv} = 2yy''' + 2y'y'' + 4y'y'' = 2yy''' + 6y'y'' \qquad y_0^{iv} = 2y_0 y_0''' + 6y_0' y_0'' = 16 + 12 = 28$

$$\therefore y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(28) + \dots$$

$$= 1 + 0.1 + 0.01 + 0.0013 + 0.00012 = 1.1114$$

$$y(0.2) = 1 + (0.2)(1) + \frac{(0.2)^2}{2!}(2) + \frac{(0.2)^3}{3!}(8) + \frac{(0.2)^4}{4!}(28) + \dots$$

$$= 1 + 0.2 + 0.04 + 0.0107 + 0.0019 = 1.2526$$

$$y(0.4) = 1 + (0.4)(1) + \frac{(0.4)^2}{2!}(2) + \frac{(0.4)^3}{3!}(8) + \frac{(0.4)^4}{4!}(28) + \dots$$

$$= 1 + 0.4 + 0.16 + 0.0853 + 0.0299 = 1.6752$$

---

④ Find the Taylor's series solution with four terms for the initial value problem, $\frac{dy}{dx} = x^3 + y$, $y(1) = 1$.

**Sol:** Given $y' = x^3 + y$, $x_0 = 1$, $y_0 = 1$

$y' = x^3 + y \qquad y_0' = x_0^3 + y_0 = 2$

$$y'' = 3x^2 + y' \qquad y_0'' = 3x_0^2 + y_0' = 5$$
$$y''' = 6x + y'' \qquad y_0''' = 6x_0 + y_0'' = 11$$
$$y^{iv} = 6 + y''' \qquad y_0^{iv} = 6 + y_0''' = 17$$

Taylor's series formula is

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$$= 1 + (x-1)(2) + \frac{(x-1)^2}{2}(5) + \frac{(x-1)^3}{6}(11) + \frac{(x-1)^4}{24}(17) + \cdots$$

$$= 1 + 2(x-1) + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{17}{24}(x-1)^4 + \cdots$$

---

(5) Using Taylor's series method with the first five terms in the expansion find $y(0.1)$ correct to 3 decimal places, given that $\frac{dy}{dx} = e^x - y^2$, $y(0) = 1$.

Sol: Given $y' = e^x - y^2$, $x_0 = 0$, $y_0 = 1$.

$$y' = e^x - y^2 \qquad y_0' = e^{x_0} - y_0^2 = 0$$
$$y'' = e^x - 2yy' \qquad y_0'' = e^{x_0} - 2y_0y_0' = 1$$
$$y''' = e^x - 2yy'' - 2y'^2 \qquad y_0''' = e^{x_0} - 2y_0y_0'' - 2y_0'^2 = 1 - 2 = -1$$
$$y^{iv} = e^x - 2yy''' - 2y'y'' - 4y'y'' \qquad y_0^{iv} = e^{x_0} - 2y_0y_0''' - 6y_0'y_0'' = 1 + 2 = 3$$
$$= e^x - 2yy''' - 6y'y''$$
$$y_0^{v} = e^{x_0} - 2y_0y_0^{iv} - 8y_0'y_0''' - 6y_0''^2 = 1 - 6 - 6 = -11$$
$$y^{v} = e^x - 2yy^{iv} - 2y'y''' - 6y'y''' - 6y''^2$$
$$= e^x - 2yy^{iv} - 8y'y''' - 6y''^2$$

Taylor's series formula is

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$$\therefore y(0.1) = 1 + (0.1)(0) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(-1) + \frac{(0.1)^4}{4!}(3) + \frac{(0.1)^5}{5!}(-11) + \cdots$$

$$= 1 + 0.005 - 0.00017 + 0.0000125 - 0.00000092 = 1.0048$$

---

<u>Taylor's series method for Simultaneous First order Differential eqns.</u>:

(1) Solve the system of eqns. $\frac{dy}{dx} = z - x^2$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$ by taking $h = 0.1$, to get $y(0.1)$ & $z(0.1)$. Here $y$ & $z$ are dependent variables & $x$ is independent.

Sol: Given $y' = z - x^2$, $z' = y + x$, $x_0 = 0$, $y_0 = 1$, $z_0 = 1$

$y' = z - x^2$ $\quad$ $y_0' = z_0 - x_0^2 = 1$ $\quad$ $z' = y + x$ $\quad$ $z_0' = y_0 + x_0 = 1$

$y'' = z' - 2x$ $\quad$ $y_0'' = z_0' - 2x_0 = 1$ $\quad$ $z'' = y' + 1$ $\quad$ $z_0'' = y_0' + 1 = 2$

$y''' = z'' - 2$ $\quad$ $y_0''' = z_0'' - 2 = 0$ $\quad$ $z''' = y''$ $\quad$ $z_0''' = y_0'' = 1$

$y^{iv} = z'''$ $\quad$ $y_0^{iv} = z_0''' = 1$ $\quad$ $z^{iv} = y'''$ $\quad$ $z_0^{iv} = y_0''' = 0$

By Taylor's series method,

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$$\therefore y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(1) + \cdots = 1.105$$

$$z(x) = z_0 + (x - x_0)z_0' + \frac{(x-x_0)^2}{2!}z_0'' + \frac{(x-x_0)^3}{3!}z_0''' + \cdots$$

$$\therefore z(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(0) + \cdots = 1.1102$$

---

<u>Solving Higher Order Linear Differential Equations:</u>

① By Taylor's series method find $y(0.1)$ given that $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$.

<u>Sol:</u> Given $y'' = y + xy'$, $x_0 = 0$, $y_0 = 1$, $y_0' = 0$.

$y'' = y + xy'$ $\qquad$ $y_0'' = y_0 + x_0 y_0' = 1$

$y''' = y' + xy'' + y' = 2y' + xy''$ $\qquad$ $y_0''' = 2y_0' + x_0 y_0'' = 0$

$y^{iv} = 2y'' + xy''' + y'' = 3y'' + xy'''$ $\qquad$ $y_0^{iv} = 3y_0'' + x_0 y_0''' = 3$

$y^{v} = 3y''' + xy^{iv} + y''' = 4y''' + xy^{iv}$ $\qquad$ $y_0^{v} = 4y_0''' + x_0 y_0^{iv} = 0$

Taylor's series formula is

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$$\therefore y(0.1) = 1 + (0.1)(0) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(3) + \frac{(0.1)^5}{5!}(0) + \cdots$$

$$= 1.005$$

---

② Evaluate the values of $y(0.1)$ & $y(0.2)$ given $y'' - x(y')^2 + y^2 = 0$, $y(0) = 1$, $y'(0) = 0$ by using Taylor's series method.

<u>Sol:</u> Given $y'' - x(y')^2 + y^2 = 0$, $x_0 = 0$, $y_0 = 1$, $y_0' = 0$

$y'' = x(y')^2 - y^2$ $\qquad$ $y_0'' = x_0(y_0')^2 - y_0^2 = -1$

$y''' = x.2y'y'' + (y')^2 - 2yy'$ $\qquad$ $y_0''' = 2x_0 y_0' y_0'' + y_0'^2 - 2y_0 y_0' = 0$

$y_0^{iv} = 2xy'y''' + y''(2xy'' + 2y') + 2y'y'' - 2yy'' - 2y'^2$ $\qquad y_0^{iv} = 2$

$= 2xy'y''' + 2xy''^2 + 4y'y'' - 2yy'' - 2y'^2$

Taylor's series formula is

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$$\therefore y(0.1) = 1 + (0.1)(0) + \frac{(0.1)^2}{2!}(-1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(2) + \cdots = 0.995$$

$$y(0.2) = 1 + (0.2)(0) + \frac{(0.2)^2}{2!}(-1) + \frac{(0.2)^3}{3!}(0) + \frac{(0.2)^4}{4!}(2) + \cdots = 0.9801$$

---

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, 2, \ldots$$

---

Problems:

① Using Euler's method find $y(0.2)$, $y(0.4)$ & $y(0.6)$ from $\frac{dy}{dx} = x + y$, $y(0) = 1$ with $h = 0.2$.

Sol: Given $\frac{dy}{dx} = f(x,y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

Euler's formula is $y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0, 1, 2, \ldots$

$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.2)f(0,1) = 1 + 0.2 = 1.2$

$\therefore y(0.2) = 1.2$

$y_2 = y_1 + hf(x_1, y_1) = 1.2 + (0.2)f(0.2, 1.2) = 1.2 + (0.2)(1.4) = 1.48$

$\therefore y(0.4) = 1.48$

$y_3 = y_2 + hf(x_2, y_2) = 1.48 + (0.2)f(0.4, 1.48) = 1.48 + (0.2)(1.88) = 1.856$

$\therefore y(0.6) = 1.856$

---

② Using Euler's method solve $y' = x + y + xy$, $y(0) = 1$, compute $y$ at $x = 0.1$, by taking $h = 0.05$.

Sol: Given $f(x,y) = x + y + xy$, $x_0 = 0$, $y_0 = 1$, $h = 0.05$, $x_1 = 0.05$, $x_2 = 0.1$

Euler's formula is $y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0, 1, 2, \ldots$

$\therefore y_1 = y_0 + hf(x_0, y_0) = 1 + (0.05)f(0,1) = 1 + (0.05)(1) = 1.05$

$\therefore y(0.05) = 1.05$

$y_2 = y_1 + hf(x_1, y_1) = 1.05 + (0.05)f(0.05, 1.05) = 1.05 + (0.05)(1.1525)$

$\therefore y_2 = y(0.1) = 1.1076$

---

③ Using Euler's method find $y(0.3)$ of $y(x)$ satisfies the initial value problem, given that $\frac{dy}{dx} = \frac{1}{2}(x^2+1)y^2$, $y(0.2) = 1.1114$.

Sol: Given $f(x,y) = \frac{1}{2}(x^2+1)y^2$, $x_0 = 0.2$, $y_0 = 1.1114$, $h = 0.1$, $x_1 = 0.3$

Euler's formula is $y_{n+1} = y_n + h f(x_n, y_n)$, $n = 0, 1, 2, \dots$

$\therefore y_1 = y_0 + h f(x_0, y_0) = 1.1114 + (0.1) f(0.2, 1.1114)$

$= 1.1114 + 0.0642 = 1.1756$

$\therefore y(0.3) = 1.1756$

Modified Euler's method:

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right], n = 0, 1, 2, \dots$$

Problems:

① Using modified Euler's method, compute $y(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}$, $y(0) = 1$.

Sol: Given $f(x,y) = y - \frac{2x}{y}$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = 0.1$

Modified Euler's formula is

$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$, $n = 0, 1, 2, \dots$

$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$

$= 1 + (0.1)\left[f\left(\frac{0.1}{2}, 1 + \frac{0.1}{2} f(0, 1)\right)\right]$

$= 1 + (0.1) f(0.05, 1.05) = 1 + (0.1)(0.9548) = 1.0955$

② Solve $y' = 1 - y$, $y(0) = 0$ by modified Euler's method.

Sol: Given $f(x,y) = 1 - y$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$.

Modified Euler's formula is

$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$, $n = 0, 1, 2, \dots$

$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$

$= 0 + (0.1)\left[f\left(\frac{0.1}{2}, 0 + \frac{0.1}{2} f(0, 0)\right)\right] = 0.1 f(0.05, 0.05)$

$= (0.1)(0.95) = 0.095$

$\therefore y(0.1) = 0.095$

$y_2 = y_1 + h\left[f\left(x_1+\frac{h}{2}, y_1+\frac{h}{2}f(x_1,y_1)\right)\right]$

$= 0.095+(0.1)\left[f\left(0.1+0.05, 0.095+0.05f(0.1, 0.095)\right)\right]$

$= 0.095+(0.1)f(0.15, 0.1403) = 0.181$ $\quad \therefore y(0.2)=0.181$

$y_3 = y_2+h\left[f\left(x_2+\frac{h}{2}, y_2+\frac{h}{2}f(x_2,y_2)\right)\right]$

$= 0.181+(0.1)\left[f\left(0.2+0.05, 0.181+0.05f(0.2, 0.181)\right)\right]$

$= 0.181+(0.1)f(0.25, 0.222) = 0.2588$

$\therefore y(0.3)=0.2588$

---

(3) Consider the initial value problem $\frac{dy}{dx} = y-x^2+1$, $y(0)=0.5$ using modified Euler's method find $y(0.2)$.

Sol: Given $f(x,y)=y-x^2+1$, $x_0=0$, $y_0=0.5$, $h=0.2$, $x_1=0.2$

Modified Euler's formula is

$y_{n+1}=y_n+h\left[f\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}f(x_n,y_n)\right)\right]$, $n=0,1,2,\dots$

$y_1=y_0+h\left[f\left(x_0+\frac{h}{2}, y_0+\frac{h}{2}f(x_0,y_0)\right)\right]$

$= 0.5+(0.2)\left[f\left(\frac{0.2}{2}, 0.5+\frac{0.2}{2}f(0, 0.5)\right)\right]$

$= 0.5+(0.2)f(0.1, 0.65) = 0.828$

$\therefore y(0.2)=0.828$

---

Fourth Order Runge-Kutta Method for solving First & Second order eqns. :

Second order R.K. Method:

$k_1 = hf(x,y)$

$k_2 = hf\left[x+\frac{h}{2}, y+\frac{k_1}{2}\right]$

$\Delta y = k_2$ where $h=\Delta x$

Third order R.K. Method:

$k_1 = hf(x,y)$

$k_2 = hf\left[x+\frac{h}{2}, y+\frac{k_1}{2}\right]$

$k_3 = hf\left[x+h, y+2k_2-k_1\right]$

& $\Delta y = \frac{1}{6}(k_1+4k_2+k_3)$

Fourth order R.K. Method:

$k_1 = hf(x,y)$

$$k_2 = hf\left[x+\frac{h}{2},\ y+\frac{k_1}{2}\right]$$
$$k_3 = hf\left[x+\frac{h}{2},\ y+\frac{k_2}{2}\right]$$
$$k_4 = hf\left[x+h,\ y+k_3\right]$$
& $\Delta y = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$ , $y(x+h) = y(x)+\Delta y$

Problems:

① Given $\frac{dy}{dx} = x^3+y$ , $y(0)=2$. Compute $y(0.2)$, $y(0.4)$ & $y(0.6)$ by R.K. method of fourth order.

Sol: Given $f(x,y) = x^3+y$ , $x_0=0$ , $y_0=2$ , $h=0.2$ , $x_1=0.2$ , $x_2=0.4$ , $x_3=0.6$.

First interval:

$k_1 = hf(x_0,y_0) = (0.2)f(0,2) = 0.4$

$k_2 = hf\left[x_0+\frac{h}{2},\ y_0+\frac{k_1}{2}\right] = (0.2)f\left(0.1,\ 2+\frac{0.4}{2}\right) = (0.2)f(0.1,\ 2.2) = 0.4402$

$k_3 = hf\left[x_0+\frac{h}{2},\ y_0+\frac{k_2}{2}\right] = (0.2)f(0.1,\ 2.2201) = 0.4442$

$k_4 = hf[x_0+h,\ y_0+k_3] = (0.2)f(0.2,\ 2.4442) = 0.4904$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.4432$

$\therefore y_1 = y_0+\Delta y = 2+0.4432 = 2.4432$

$\therefore y(0.2) = 2.4432$

Second interval:

$k_1 = hf(x_1,y_1) = (0.2)f(0.2,\ 2.4432) = 0.4902$

$k_2 = hf\left[x_1+\frac{h}{2},\ y_1+\frac{k_1}{2}\right] = (0.2)f(0.3,\ 2.6883) = 0.5431$

$k_3 = hf\left[x_1+\frac{h}{2},\ y_1+\frac{k_2}{2}\right] = (0.2)f(0.3,\ 2.7148) = 0.5484$

$k_4 = hf[x_1+h,\ y_1+k_3] = (0.2)f(0.4,\ 2.9916) = 0.6111$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.5474$

$\therefore y_2 = y_1+\Delta y = 2.4432+0.5474 = 2.9906$

$\therefore y(0.4) = 2.9906$

Third interval:

$k_1 = hf(x_2,y_2) = (0.2)f(0.4,\ 2.9906) = 0.6109$

$k_2 = hf\left[x_2+\frac{h}{2},\ y_2+\frac{k_1}{2}\right] = (0.2)f(0.5,\ 3.2961) = 0.6842$

$k_3 = hf\left[x_2+\frac{h}{2},\ y_2+\frac{k_2}{2}\right] = (0.2)f(0.5,\ 3.3327) = 0.6915$

$k_4 = hf[x_2+h,\ y_2+k_3] = (0.2)f(0.6,\ 3.6821) = 0.7796$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.6903$

$\therefore y_3 = y_2 + \Delta y = 2.9906 + 0.6903 = 3.6809$

$\therefore y(0.6) = 3.6809.$

② Using R.K. Method of 4th order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$.

Sol: Given $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0$, $y_0=1$, $h=0.2$, $x_1=0.2$

$k_1 = hf(x_0, y_0) = (0.2)f(0,1) = (0.2)(1) = 0.2$

$k_2 = hf\left[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right] = (0.2)f(0.1, 1.1) = 1.1967$

$k_3 = hf\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right] = (0.2)f(0.1, 1.5984) = 0.1984$

$k_4 = hf\left[x_0+h, y_0+k_3\right] = (0.2)f(0.2, 1.1984) = 0.1892$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = \frac{1}{6}(3.1794) = 0.5299$

$\therefore y_1 = y_0 + \Delta y = 1 + 0.5299 = 1.5299$

$\therefore y(0.2) = 1.5299$

③ Find $y(0.8)$ given that $y' = y - x^2$, $y(0.6) = 1.7379$ by using R.K. method of 4th order. Take $h=0.1$

Sol: Given $f(x,y) = y - x^2$, $x_0=0.6$, $y_0=1.7379$, $h=0.1$, $x_1=0.7$, $x_2=0.8$

First interval:

$k_1 = hf(x_0, y_0) = (0.1)f(0.6, 1.7379) = 0.1378$

$k_2 = hf\left[x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right] = (0.1)f(0.65, 1.8068) = 0.1384$

$k_3 = hf\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right] = (0.1)f(0.65, 1.8071) = 0.1385$

$k_4 = hf\left[x_0+h, y_0+k_3\right] = (0.1)f(0.7, 1.8764) = 0.1386$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = \frac{1}{6}(0.8302) = 0.1384$

$\therefore y_1 = y_0 + \Delta y = 1.7379 + 0.1384 = 1.8763$

$\therefore y(0.7) = 1.8763$

Second interval:

$k_1 = hf(x_1, y_1) = (0.1)f(0.7, 1.8763) = 0.1386$

$k_2 = hf\left[x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right] = (0.1)f(0.75, 1.9456) = 0.1383$

$k_3 = hf\left[x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right] = (0.1)f(0.75, 1.9455) = 0.1383$

$k_4 = hf\left[x_1+h, y_1+k_3\right] = (0.1)f(0.8, 2.0146) = 0.1375$

$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = \frac{1}{6}(0.8293) = 0.1382$

$y_2 = y_1 + \Delta y = 1.8763 + 0.1382 = 2.0145$

$\therefore y(0.8) = 2.0145$

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R.K. Method for Simultaneous First order Differential Equation:

Solving the eqn. $\frac{dy}{dx} = f_1(x,y,z)$ & $\frac{dz}{dx} = f_2(x,y,z)$ with the initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$. (Here $x$ is independent variable while $y$ & $z$ are dependent variable).

$k_1 = hf_1(x_0, y_0, z_0)$

$k_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$k_3 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$k_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + l_3]$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_1 = y_0 + \Delta y$

$l_1 = hf_2(x_0, y_0, z_0)$

$l_2 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$l_3 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$l_4 = hf_2[x_0 + h, y_0 + k_3, z_0 + l_3]$

$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$z_1 = z_0 + \Delta z$

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④ Solving the system of differential eqns. $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dx} = -xy$ for $x = 0.3$ using 4th order R.K. Method, the initial values are $x = 0, y = 0, z = 1$.

Sol: Given $f_1(x,y,z) = \frac{dy}{dx} = xz + 1$, $f_2(x,y,z) = \frac{dz}{dx} = -xy$, $x_0 = 0$, $y_0 = 0$, $z_0 = 1$, $h = 0.3$

$f_1(x,y,z) = xz + 1$

$k_1 = hf_1(x_0, y_0, z_0) = (0.3) f_1(0,0,1)$

$= 0.3$

$k_2 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= (0.3) f_1[0.15, 0.15, 1] = 0.345$

$k_3 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$= (0.3) f_1(0.15, 0.1725, 0.9966)$

$= 0.3448$

$k_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + l_3]$

$= (0.3) f_1(0.3, 0.3448, 0.9922)$

$= 0.3893$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(2.0689) = 0.3448$

$f_2(x,y,z) = -xy$

$l_1 = hf_2(x_0, y_0, z_0) = (0.3) f_2(0,0,1)$

$= 0$

$l_2 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= (0.3) f_2(0.15, 0.15, 1) = -0.0068$

$l_3 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$

$= (0.3) f_2(0.15, 0.1725, 0.9966)$

$= -0.0078$

$l_4 = hf_2[x_0 + h, y_0 + k_3, z_0 + l_3]$

$= (0.3) f_2(0.3, 0.3448, 0.9922)$

$= -0.031$

$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = \frac{1}{6}(-0.0602)$

$= -0.01$

$y_1 = y_0 + \Delta y = 0.3448$  $z_1 = z_0 + \Delta z = 1 - 0.01 = 0.99$

Hence $y(0.3) = 0.3448$ & $z(0.3) = 0.99$.

## R.K. Method for Second Order Differential Equations:

(5) Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ & $y'(0) = -0.6$ using $4^{th}$ order R.K. method, find $y(0.2)$.

Sol: Let $t = x$

$y'' = 2y' - 2y + e^{2x}\sin x$ , $x_0 = 0$, $y_0 = -0.4$, $y_0' = -0.6$, $h = 0.2$

Setting $y' = z$ the eqn. becomes, $z' = 2z - 2y + e^{2x}\sin x$

$f_1(x,y,z) = \frac{dy}{dx} = z$  $f_2(x,y,z) = \frac{dz}{dx} = 2z - 2y + e^{2x}\sin x$

Given: $x_0 = 0$, $y_0 = -0.4$, $y_0' = z_0 = -0.6$, $h = 0.2$

$k_1 = hf_1(x_0, y_0, z_0) = (0.2)f_1(0, -0.4, -0.6)$
$= (0.2)(-0.6) = -0.12$

$l_1 = hf_2(x_0, y_0, z_0) = (0.2)f_2(0, -0.4, -0.6)$
$= (0.2)(-0.4) = -0.08$

$k_2 = hf_1(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$
$= (0.2)f_1(0.1, -0.46, -0.64) = -0.128$

$l_2 = hf_2[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}]$
$= (0.2)f_2(0.1, -0.46, -0.64) = -0.0476$

$k_3 = hf_1(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$
$= (0.2)f_1(0.1, -0.464, -0.6238) = -0.1248$

$l_3 = hf_2[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}]$
$= (0.2)f_2(0.1, -0.464, -0.6238) = -0.0395$

$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= (0.2)f_1(0.2, -0.5248, -0.6395) = -0.1279$

$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$
$= (0.2)f_2(0.2, -0.5248, -0.6395)$
$= 0.0134$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.1256$

$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.0401$

$y_1 = y_0 + \Delta y = -0.4 - 0.1256 = -0.5256$

$\therefore y(0.2) = -0.5256$

## Milne's Predictor And Corrector Methods:

In the Milne's method, we suppose that four equispaced starting values of $y$ are known, at the pts $x_n, x_{n-1}, x_{n-2}$ & $x_{n-3}$.

Milne's predictor & Corrector formulae:

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

Problems:

① Given $\frac{dy}{dx} = x^3+y$, $y(0)=2$. The values of $y(0.2)=2.073$, $y(0.4)=2.4524$, $y(0.6)=3.023$ are got by R.K. method of 4th order. Find $y(0.8)$ by Milne's predictor-corrector methods taking $h=0.2$.

Sol: Here $x_0=0$ $\quad y_0=2$ $\quad h=0.2$

$x_1=0.2$ $\quad y_1=2.073$

$x_2=0.4$ $\quad y_2=2.452$

$x_3=0.6$ $\quad y_3=3.023$

$y'=x^3+y$ —①

Milne's predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right) \quad \text{—②}$$

From ①, $y'_1 = x_1^3 + y_1 = 2.081$

$y'_2 = x_2^3 + y_2 = 2.516$

$y'_3 = x_3^3 + y_3 = 3.239$

$$\therefore ② \Rightarrow y_{4,p} = 2 + \frac{4(0.2)}{3}\left(2(2.081) - 2.516 + 2(3.239)\right)$$

$$= 2 + 2.1664 = 4.1664$$

Milne's corrector formula is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$$

$$y_{4,c} = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right) \quad \text{—③}$$

$$y'_4 = x_4^3 + y_4 = 4.6784$$

$$\therefore ③ \Rightarrow y_{4,c} = 2.452 + \frac{(0.2)}{3}\left(2.516 + 4(3.239) + 4.6784\right)$$

$$= 2.452 + 1.3434 = 3.7954$$

$\therefore y(0.8) = 3.7954$

② Solve $y' = x - y^2$, $0 \le x \le 1$, $y(0)=0$, $y(0.2)=0.02$, $y(0.4)=0.0795$, $y(0.6)=0.1762$ by Milne's method to find $y(0.8)$ & $y(1)$.

Sol: Here $x_0=0$ $\quad y_0=0$

$x_1=0.2$ $\quad y_1=0.02$

$x_2=0.4$ $\quad y_2=0.0795$

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$x_3 = 0.6 \quad y_3 = 0.1762$

$x_4 = 0.8, \ x_5 = 1, \ h = 0.2$

Given $y' = x - y^2$ — ①

Milne's predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$y_{4,p} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) \text{ — ②}$$

From ①, $y'_1 = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$

$y'_2 = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3937$

$y'_3 = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.569$

$\therefore$ ② $\Rightarrow y_{4,p} = 0 + \frac{4(0.2)}{3}(2(0.1996) - 0.3937 + 2(0.569))$

$= \frac{4(0.2)}{3}(1.1435) = 0.3049$

Milne's corrector formula is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

$$y_{4,c} = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'_4) \text{ — ③}$$

$y'_4 = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

$\therefore$ ③ $\Rightarrow y_{4,c} = 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.569) + 0.707)$

$= 0.0795 + 0.2251 = 0.3046$

$\therefore y(0.8) = 0.3046 = y_4$

To find $y(1)$, $y_{5,p} = y_1 + \frac{4h}{3}(2y'_2 - y'_3 + 2y'_4)$

$y'_4 = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

$\therefore y_{5,p} = 0.02 + \frac{4(0.2)}{3}(2(0.3937) - 0.569 + 2(0.7072))$

$= 0.02 + \frac{4(0.2)}{3}(1.6328) = 0.4554$

$y_{5,c} = y_3 + \frac{h}{3}(y'_3 + 4y'_4 + y'_5)$

$y'_5 = x_5 - y_5^2 = 1 - (0.4554)^2 = 0.7926$

$\therefore y_{5,c} = 0.1762 + \frac{0.2}{3}(0.569 + 4(0.7072) + 0.7926)$

$= 0.1762 + 0.2794 = 0.4556$

$\therefore y(1) = 0.4556$

⑧ Given $y' = 1-y$ & $y(0)=0$, find (i) $y(0.1)$ by Euler's method (ii) $y(0.2)$ by modified Euler's method (iii) $y(0.4)$ by Milne's method.

Sol: Given $y'=1-y$, $x_0=0$, $y_0=0$, $x_1=0.1$, $x_2=0.2$, $x_3=0.3$, $x_4=0.4$, $h=0.1$

(i) Euler's method: $y_{n+1} = y_n + h f(x_n, y_n)$

$$y_1 = y_0 + h f(x_0, y_0) = 0 + (0.1) f(0,0) = 0.1$$

$\therefore y(0.1) = 0.1$

(ii) Modified Euler's method: $y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$

$$y_2 = y_1 + h\left[f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right)\right]$$

$$= 0.1 + (0.1)\left[f\left(0.1 + \frac{0.1}{2}, 0.1 + \frac{0.1}{2} f(0.1, 0.1)\right)\right]$$

$$= 0.1 + (0.1) f(0.15, 0.145) = 0.1855$$

$\therefore y(0.2) = 0.1855$

By using Euler's method, $y_3 = y_2 + h f(x_2, y_2)$

$$= 0.1855 + (0.1) f(0.2, 0.1855)$$

$$= 0.267$$

$\therefore y(0.3) = 0.267$

(iii) Milne's method:

Here $x_0 = 0$ $\quad y_0 = 0$ $\quad h = 0.1$

$x_1 = 0.1$ $\quad y_1 = 0.1$

$x_2 = 0.2$ $\quad y_2 = 0.1855$

$x_3 = 0.3$ $\quad y_3 = 0.267$

Milne's predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

$$y'_1 = 1 - y_1 = 1 - 0.1 = 0.9$$

$$y'_2 = 1 - y_2 = 1 - 0.1855 = 0.8145$$

$$y'_3 = 1 - y_3 = 1 - 0.267 = 0.733$$

$$\therefore y_{4,p} = 0 + \frac{4(0.1)}{3}\left(2(0.9) - 0.8145 + 2(0.733)\right) = 0.3269$$

Milne's corrector formula is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$$

$$y_{4,c} = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right)$$

$\Rightarrow y_2 - 1.9375 y_3 = -1.9531$ —— (6)

(5) $\times 1.9375 \Rightarrow 1.9375 y_1 - 3.7539 y_2 + 1.9375 y_3 = 0.0606$ —— (7)

(4)+(7) $\Rightarrow -2.7539 y_2 + 1.9375 y_3 = 0.0762$ —— (8)

(6)+(8) $\Rightarrow -1.7539 y_2 = -1.8769 \Rightarrow y_2 = 1.0701$

Subst. $y_2$ in (4), $-1.9375 y_1 + 1.0701 = 0.0156$

$\Rightarrow y_1 = 0.5443$

Subst. $y_2$ in (6), $1.0701 - 1.9375 y_3 = -1.9531$

$\Rightarrow y_3 = 1.5604$

Hence $y(0.25) = 0.5443$, $y(0.5) = 1.0701$, $y(0.75) = 1.5604$

(2) Solve the eqn. $y''(x) - xy(x) = 0$ for $y(x_i)$, $x_i = 0, \frac{1}{3}, \frac{2}{3}$, given that $y(0) + y'(0) = 1$ & $y(1) = 1$.

<u>Sol:</u> The given eqn. can be written as

$$y_i'' - x_i y_i = 0 \quad (1)$$

Using the central difference approximation, we have

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$\therefore$ (1) becomes, $\dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - x_i y_i = 0$

$\Rightarrow y_{i-1} - (2 + x_i h^2) y_i + y_{i+1} = 0$

$\Rightarrow y_{i-1} - (2 + \frac{1}{9} x_i) y_i + y_{i+1} = 0$ —— (1) $(\because h = 1/3)$

Put $i = 0, 1, 2$ in (1), we have

$y_{-1} - (2 + \frac{1}{9} x_0) y_0 + y_1 = 0 \Rightarrow y_{-1} - 2y_0 + y_1 = 0$ —— (2)

$y_0 - (2 + \frac{1}{9} x_1) y_1 + y_2 = 0 \Rightarrow y_0 - 2.037 y_1 + y_2 = 0$ —— (3)

$y_1 - (2 + \frac{1}{9} x_2) y_2 + y_3 = 0 \Rightarrow y_1 - 2.0741 y_2 + y_3 = 0$ —— (4)

Since $y_i' = \dfrac{y_{i+1} - y_{i-1}}{2h}$, the first boundary condition becomes

$y_0 + y_0' = 1 \Rightarrow y_0 + \dfrac{y_1 - y_{-1}}{2/3} = 1 \Rightarrow 2y_0 + 3(y_1 - y_{-1}) = 2$

$\Rightarrow 3y_{-1} = 2y_0 + 3y_1 - 2$

$\Rightarrow y_{-1} = \frac{1}{3}(2y_0 + 3y_1 - 2)$

The second boundary condition is $y_3 = 1$

Subst. $y_{-1}$ in (2), $3y_1 - 2y_0 - 1 = 0$ —— (5)

Subst. $y_3$ in (4), $y_1 - 2.0741 y_2 + 1 = 0$ —— (6)

## Milne's Predictor & Corrector Methods For Solving First Order Differential Equations :

Suppose $\frac{dy}{dx} = f(x,y)$ is a given diff. eqn

with values $y(x_0) = y_0$ (Boundary condition)

$$\left.\begin{aligned} y(x_1) &= y_1 \\ y(x_2) &= y_2 \\ y(x_3) &= y_3 \end{aligned}\right\} \text{Soln got by R-K method (Or) Taylors, (Or) Eulers}$$

then we predict the soln $y_{n+1}$ by using the Milnes' predictor formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

And the Milnes' corrector formula is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

Here $y_{n+1}$ is obtained in Milne's predictor formula.

Problems

① Given $\frac{dy}{dx} = x^3+y$, $y(0) = 2$. The values of $y(0.2) = 2.073$, $y(0.4) = 2.452$, $y(0.6) = 3.023$ are got by R-K method of fourth order. Find $y(0.8)$ by Milnes predictor-corrector method by taking $h=0.2$.

Soln:

Here $\frac{dy}{dx} = x^3+y = f(x,y)$.

Given $x_0 = 0$, $y_0 = 2$

$x_1 = 0.2$, $y_1 = 2.073$

$x_2 = 0.4$, $y_2 = 2.452$

$x_3 = 0.6$, $y_3 = 3.023$.

$y' = x^3+y$.

$y_1' = x_1^3 + y_1 = (0.2)^3 + 2.073 = 2.081$

$y_2' = x_2^3 + y_2 = (0.4)^3 + 2.452 = 2.516$

$y_3' = x_3^2 + y_3 = (0.6)^3 + 3.023 = 3.239$.

By Milne's predictor Formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' - y_{n-1}' + 2y_n'\right]$$

Put $n=3$

$$y_{4,p} = y_0 + \frac{4(0.2)}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 2 + \frac{4(0.2)}{3}\left[2(2.081) - 2.516 + 2(3.239)\right]$$

$$= 2 + \frac{0.8}{3}\left[8.124\right]$$

$$y_{4,p} = 4.1664.$$

(i.e) We predict $y_4$ as $4.1664$.

Using Milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right].$$

$$y_4' = x_4^3 + y_4 = (0.8)^3 + 4.1664 = 4.6784.$$

$$\therefore\ y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 2.452 + \frac{0.2}{3}\left[2.516 + 4(3.239) + 4.6784\right]$$

$$= 2.452 + \frac{0.2}{3}\left[20.1504\right]$$

$$= 3.79536.$$

$\therefore$ Corrected value of $y$ at $(0.8)$ is $3.79536$.

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(2) Using Milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$.

Soln

Given $y' = \dfrac{2 - y^2}{5x} = f(x, y)$.

Also given

$x_0 = 4 \qquad y_0 = 1$

$x_1 = 4.1 \qquad y_1 = 1.0049$

$x_2 = 4.2 \qquad y_2 = 1.0097$

$x_3 = 4.3 \qquad y_3 = 1.0143$

$$y' = \frac{2 - y^2}{5x}.$$

$$y_1' = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.0493$$

$$y_2' = \frac{2 - y_2^2}{5(x_2)} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.0467.$$

$$y_3' = \frac{2 - y_3^2}{5(x_3)} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.0452.$$

By Milne's Predictor Formula

$$y_{n+1, p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

Put $n = 3$

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_{4,p} = 1 + \frac{4(0.1)}{3}\left[2(0.0493) - 0.0467 + 2(0.0452)\right]$$

$$= 1.01897.$$

(c) We predict $y_4$ as $1.01897$.

By Milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right]$$

Put $n=3$

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$y_4' = \frac{2 - (y_4)^2}{5(x_4)} = \frac{2 - (1.01897)^2}{5(4.4)}$$

$$= 0.0437.$$

$$y_{4,c} = 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0452) + 0.0437\right]$$

$$y_{4,c} = 1.01874.$$

$\therefore$ Corrected value of $y_4$ is $1.01874$.